

A GEOMETRIC REALISATION OF 0-SCHUR AND 0-HECKE ALGEBRAS

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ABSTRACT. We define a new product on orbits of pairs of flags in a vector space, using open orbits in certain varieties of pairs of flags. This new product defines an associative \mathbb{Z} -algebra, denoted by $G(n, r)$. We show that $G(n, r)$ is a geometric realisation of the 0-Schur algebra $S_0(n, r)$ over \mathbb{Z} , which is the q -Schur algebra $S_q(n, r)$ at $q = 0$. We view a pair of flags as a pair of projective resolutions for a quiver of type \mathbb{A} with linear orientation, and study q -Schur algebras from this point of view. This allows us to understand the relation between q -Schur algebras and Hall algebras and construct bases of q -Schur algebras, which are used in the proof of the main results. Using the geometric realisation, we construct idempotents and multiplicative bases for 0-Schur algebras. We also give a geometric realisation of 0-Hecke algebras and a presentation of the q -Schur algebra over a base ring where q is not invertible.

INTRODUCTION

Let k be a finite or an algebraically closed field. When k is finite, we denote by q the number of elements in k . Let $n \geq 1$ and $r \geq 1$ be integers and let \mathcal{F} denote the variety of partial r -step flags in an n -dimensional vector space V . The general linear group $\mathrm{GL}(V)$ acts on \mathcal{F} by change of basis on V and the orbits under this action are denoted by $\mathcal{F}_{\underline{d}}$, where \underline{d} is a decomposition of the integer n into r parts.

Beilinson, Lusztig and MacPherson [1] (see also Du [11] and Green [16]) construct the q -Schur algebra $S_q(n, r)$ on the basis of $\mathrm{GL}(V)$ -orbits in $\mathcal{F} \times \mathcal{F}$, where $\mathrm{GL}(V)$ acts diagonally on $\mathcal{F} \times \mathcal{F}$ by change of basis on V . This algebra can be defined using the pair of maps

$$\begin{array}{ccc} \mathcal{F} \times \mathcal{F} \times \mathcal{F} & \xrightarrow{\Delta} & (\mathcal{F} \times \mathcal{F}) \times (\mathcal{F} \times \mathcal{F}) \\ \downarrow \pi & & \\ \mathcal{F} \times \mathcal{F} & & \end{array}$$

where $\Delta(f, f', f'') = ((f, f'), (f', f''))$ and $\pi(f, f', f'') = (f, f'')$, and the structure constant in front of an orbit $[g, g'] = \mathrm{GL}(V)(g, g')$ in the product of $[f, f']$ and $[f', f'']$ is the number of elements in the set

$$\pi^{-1}(g, g') \cap \Delta^{-1}([f, f'] \times [f', f'']).$$

The structure constants are given by polynomials evaluated at q for any finite field. Beilinson, Lusztig and MacPherson construct a limit of these finite dimensional algebras to give a geometric realisation of the quantised enveloping algebra of gl_n .

In this paper we take the point of view that a pair of flags is a pair of projective resolutions for a quiver of type \mathbb{A} with linear orientation. We show that a pair of flags and its corresponding pair of projective resolutions uniquely determine each other. We then construct bases and prove results that will be used in the subsequent parts of the paper.

The main goal of this paper is to give a geometric realisation of 0-Schur algebras, which are q -Schur algebras at $q = 0$. We define a new algebra $G(n, r)$ on the same basis as $S_q(n, r)$ by defining the product of $[f, f']$ and $[f', f'']$ to be the unique open orbit (see Section 6) in

$$\pi \Delta^{-1}([f, f'] \times [f', f'']),$$

and obtain the following main result.

Theorem 1. *As \mathbb{Z} -algebras, $G(n, r)$ is isomorphic to $S_q(n, r)_{q=0}$.*

We remark that the definition of the new product is similar to the one defined by Reineke [21] for Hall algebras and the main result generalises the main result in [24].

To prove Theorem 1, we first give a presentation of $G(n, r)$ using quivers and relations and then use the presentation to prove the isomorphism in the theorem. We remark that Deng and Yang [6] have independently given a similar presentation for $S_0(n, r)$, using a different approach.

Among applications of the geometric realisation, we obtain a multiplicative basis for $S_0(n, r)$ and using open orbits we construct a block in $S_0(n, r)$ isomorphic to a matrix algebra and families of idempotents. Also, we give a presentation of $S_q(n, r)$ over a base ring where q is not invertible. In the special case $n = r$, we obtain a geometric realisation of the 0-Hecke algebra $H_0(n)$.

The paper is organised as follows. In Section 1 we recall the construction of Beilinson, Lusztig and MacPherson and in Section 2 we describe orbits in $\mathcal{F} \times \mathcal{F}$ using representations of linear quivers of type A. In Section 3 we recall the definition of the positive and negative parts of the q -Schur algebras and their relationship to the Hall algebra, and use this description to construct a basis of the q -Schur algebra in Section 4. In Section 5 we give a description of quantised Schur algebras using quivers and relations. We define the generic algebra in Section 6 and show that it is isomorphic to the 0-Schur-algebra in Section 7. In Section 8 we consider the degeneration order of orbits in $\mathcal{F} \times \mathcal{F}$, and use open orbits to construct idempotents for the 0-Schur algebra in Section 9. Finally, we discuss 0-Hecke algebras in Section 10.

1. FLAG VARIETIES AND q -SCHUR ALGEBRAS

In this section we fix notation and recall some definitions and results of Beilinson, Lusztig and MacPherson [1] on q -Schur algebras. We also recall the definition of the 0-Schur algebra.

Let $n, r \geq 1$ be integers and V an r -dimensional vector space over a field k . Denote by \mathcal{F} the set of all n -steps flags

$$f : \{0\} = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$$

in V . The general linear group $\mathrm{GL}(V)$ acts on \mathcal{F} by change of basis on V .

For any $f \in \mathcal{F}$, define $d_i = \dim V_i - \dim V_{i-1}$ for $i = 1, \dots, n$. Then

$$\underline{d} = d_1 + \cdots + d_n$$

is a decomposition of r into n parts. Two flags f and g are isomorphic, i.e. they are in the same $\mathrm{GL}(V)$ -orbit, if and only if they have the same decomposition. Let $D(n, r)$ denote the set of all decompositions of r in n parts, and let $\mathcal{F}_{\underline{d}} \subseteq \mathcal{F}$ denote the orbit corresponding to $\underline{d} \in D(n, r)$.

Let $\mathrm{GL}(V)$ act diagonally on $\mathcal{F} \times \mathcal{F}$, i.e. by change of basis on V . Given a pair of flags $(f, f') \in \mathcal{F} \times \mathcal{F}$,

$$f : \{0\} = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$$

and

$$f' : \{0\} = V'_0 \subseteq V'_1 \subseteq \cdots \subseteq V'_n = V,$$

define a matrix $A = A(f, f') = (A_{ij})$, with

$$A_{ij} = \dim(V_{i-1} + V_i \cap V'_j) - \dim(V_{i-1} + V_i \cap V'_{j-1}).$$

This defines a bijection between the $\mathrm{GL}(V)$ -orbits in $\mathcal{F} \times \mathcal{F}$ and matrices of non-negative integers with entries which sum to r . We denote the $\mathrm{GL}(V)$ -orbit of (f, f') by $[f, f']$ and by e_A if we want to emphasize the matrix $A = A(f, f')$. Two pairs of flags (f, f') and (g, g') are isomorphic if they belong to the same $\mathrm{GL}(V)$ -orbit and we write $(f, f') \simeq (g, g')$ in this case.

Let $e_A, e_{A'}, e_{A''} \in \mathcal{F} \times \mathcal{F} / \mathrm{GL}(V)$ and $(f_1, f_2) \in e_{A''}$. Let

$$S(A, A', A'') = \{f \in \mathcal{F} \mid (f_1, f) \in e_A, (f, f_2) \in e_{A'}\}.$$

Following Proposition 1.1 in [1], there exists a polynomial $g_{A, A', A''} \in \mathbb{Z}[q]$, such that

$$g_{A, A', A''}(q) = |S(A, A', A'')|,$$

for all finite fields. The projection $\mathcal{F} \times \mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F}$ onto the middle factor maps $\Delta^{-1}(e_A \times e_{A'}) \cap \pi^{-1}(f_1, f_2)$ bijectively onto $S(A, A', A'')$, and so these two sets have the same cardinality.

Recall [1] that the q -Schur algebra $S_q(n, r)$ is the free $\mathbb{Z}[q]$ -module with basis $\mathcal{F} \times \mathcal{F} / \mathrm{GL}(V)$, and associative multiplication given by

$$e_A e_{A'} = \sum_{e_{A''} \in \mathcal{F} \times \mathcal{F} / \mathrm{GL}(V)} g_{A, A', A''} e_{A''}.$$

Although, in general it is difficult to compute the polynomial $g_{A, A', A''}$, the following lemma from [1], dealing with special A and A' , gives clear multiplication rules. Also, Deng and Yang give a recursive formula of $g_{A, A', A''}$ using Hall polynomials [14].

Let

$$[m] = \frac{q^m - 1}{q - 1} = q^{m-1} + \cdots + q + 1$$

for $m \in \mathbb{N}$ and let $E_{i,j}$ denote the (i,j) 'th elementary matrix.

Lemma 1.1. [1] *Assume that $1 \leq h < n$. Let $e_A \subseteq \mathcal{F}_{\underline{e}} \times \mathcal{F}_{\underline{f}}$. Assume that $e_B \subseteq \mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{e}}$ and $e_C \subseteq \mathcal{F}_{\underline{d}'} \times \mathcal{F}_{\underline{e}}$ such that $B - E_{h,h+1}$, $C - E_{h+1,h}$ are diagonal matrices. Then the following multiplication formulæ hold in $S_q(n, r)$,*

$$\begin{aligned} e_B e_A &= \sum_{\{p | A_{h+1,p} > 0\}} q^{\sum_{j>p} A_{h,j}} [A_{h,p} + 1] e_X, \\ e_C e_A &= \sum_{\{p | A_{h,p} > 0\}} q^{\sum_{j<p} A_{h+1,j}} [A_{h+1,p} + 1] e_Y, \end{aligned}$$

where $X = A + E_{h,p} - E_{h+1,p}$ and $Y = A - E_{h,p} + E_{h+1,p}$.

The classical Schur algebra

$$S(n, r) = S(n, r) \otimes_{\mathbb{Z}[q]} \mathbb{Z}[q]/(q-1)$$

is obtained by evaluating the structure constants $g_{A,A',A''}$ at $q = 1$, and the 0-Schur algebra

$$S_0(n, r) = S_q(n, r) \otimes_{\mathbb{Z}[q]} \mathbb{Z}[q]/(q)$$

is obtained by evaluating the structure constants $g_{A,A',A''}$ at $q = 0$.

2. REPRESENTATIONS OF LINEAR QUIVERS

In this section we describe orbits of pairs of flags using representations of a linear quiver $\Lambda = \Lambda(n)$ of type \mathbb{A}_n ,

$$\Lambda(n) : 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n.$$

A representation X of Λ consists of vector spaces X_i , $i = 1, \dots, n$, and linear maps $X_\alpha : X_i \rightarrow X_{i+1}$, $\alpha : i \rightarrow i+1$, $i = 1, \dots, n-1$. A homomorphism $h : X \rightarrow Y$ between two representations X and Y is a collection of linear maps $h_i : X_i \rightarrow Y_i$, satisfying $h_{i+1} X_\alpha = Y_\alpha h_i$. A homomorphism h is an isomorphism if h_i is a linear isomorphism for all $i = 1, \dots, n$, and we write $X \simeq Y$ if X and Y are isomorphic. A direct sum of representations is a direct sum of vector spaces and maps. A nonzero representation is indecomposable if it is not isomorphic to a direct sum of non-zero representations.

The dimension vector of X is denoted by $\underline{\dim} X = (\dim X_i)_i$. If we fix each vector space X_i we may parameterise representations of Λ by the vector space of all maps $(X_\alpha)_\alpha$ on which $\prod_i \text{GL}(X_i)$ acts by change of basis, such that the orbits correspond to isomorphism classes of representations with dimension vector $\underline{\dim} X$.

Let M_{ij} , for $j \geq i$, be the indecomposable representation supported on the interval of vertices $[i, j] = \{i, \dots, j\}$, with vector spaces in the support equal to k and all non-zero maps equal to the identity. Any representation M decomposes uniquely, up to isomorphism, as

$$M \simeq \bigoplus_{i,j} (M_{ij})^{d_{ij}}$$

for non-negative integers d_{ij} . For each vertex i , let $S_i = M_{ii}$ and $P_i = M_{in}$ be the simple and indecomposable projective representation at i , respectively. A representation P is projective if and only if each map P_α is injective.

For each decomposition $\underline{d} \in D(n, r)$, let $P(\underline{d})$ be the projective representation defined by

$$P(\underline{d}) = \bigoplus_i (P_i)^{d_i}, P(\underline{d})_n = V.$$

By taking images of the maps $P(\underline{d})_i \rightarrow P(\underline{d})_n = V$ we get a n -step flag in $\mathcal{F}_{\underline{d}}$. We will view any projective representation $P(\underline{d})$ as a flag in $\mathcal{F}_{\underline{d}}$.

An n -step flag in V is a projective representation of Λ , with maps equal to the inclusions $V_i \subseteq V_{i+1}$. Two flags are isomorphic if and only if they are isomorphic as representations. If f is a flag in U and f' is a flag in U' then $f \oplus f'$ denotes the flag in $V = U \oplus U'$ with vector space at each vertex i equal to $U_i \oplus U'_i$.

A pair of flags (g, f) with $g \subseteq f$ can be viewed as a projective resolution

$$0 \rightarrow g \rightarrow f \rightarrow f/g \rightarrow 0.$$

If $(f_1, f_2), (f'_1, f'_2) \in \mathcal{F} \times \mathcal{F}$, with $f_1 \subseteq f_2$ and $f'_1 \subseteq f'_2$, then $(f_1, f_2) \simeq (f'_1, f'_2)$ if and only if $f_2/f_1 \simeq f'_2/f'_1$ and $f_2 \simeq f'_2$. This fact generalises to arbitrary pairs, and this is the main lemma of this section.

Lemma 2.1. *Let $(f_1, f_2), (f'_1, f'_2) \in \mathcal{F} \times \mathcal{F}$. The following are equivalent.*

- i) $(f_1, f_2) \simeq (f'_1, f'_2)$.
- ii) $(f_1 + f_2)/f_i \simeq (f'_1 + f'_2)/f'_i$ for $i = 1, 2$, and $f_1 + f_2 \simeq f'_1 + f'_2$.
- iii) $(f_i, f_1 + f_2) \simeq (f'_i, f'_1 + f'_2)$ for $i = 1, 2$.

Proof. The implication from i) to ii) is trivial.

We prove that ii) implies i). By ii), $f_i/(f_1 \cap f_2) \simeq f'_i/(f'_1 \cap f'_2)$, and $f_1 \cap f_2 \simeq f'_1 \cap f'_2$.

Let $g_i : f_i/f_1 \cap f_2 \rightarrow f'_i/f'_1 \cap f'_2$ be isomorphisms. Consider the following diagram,

$$\begin{array}{ccc} f_1 + f_2 & \xrightarrow{\pi} & (f_1/f_1 \cap f_2) \oplus (f_2/f_1 \cap f_2) \\ \downarrow \exists h & & \downarrow \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \\ f'_1 + f'_2 & \xrightarrow{\pi'} & (f'_1/f'_1 \cap f'_2) \oplus (f'_2/f'_1 \cap f'_2), \end{array}$$

where π and π' are natural projections. Since $f_1 + f_2$ and $f'_1 + f'_2$ are isomorphic projective representations, there is an isomorphism h such that the above diagram commutes. Thus $h(f_1) \subseteq (\pi')^{-1}(f'_1/f'_1 \cap f'_2) = f'_1$. Therefore $h(f_1) = f'_1$. Similarly $h(f_2) = f'_2$. Hence (f_1, f_2) and (f'_1, f'_2) are in the same orbit. This proves i)

iii) is a reformulation of ii). This finishes the proof. \square

We mention that it is possible to restate the lemma by replacing the inclusions $f_i \subseteq f_1 + f_2$ by the inclusions $f_1 \cap f_2 \subseteq f_i$. Also, the condition $f_1 + f_2 \simeq f'_1 + f'_2$ in part ii) can be replaced by $f_1 \simeq f'_1$ and $f_2 \simeq f'_2$.

We give some consequences of the lemma. The lemma shows that pairs of flags in $\mathcal{F} \times \mathcal{F}$ are determined up to isomorphism by triples $(\underline{d}, [M], [N])$, where $\underline{d} \in D(n, r)$ and $[M], [N]$ are isomorphism classes of representations of Λ with a surjection $P(\underline{d}) \rightarrow M \oplus N$. Conversely, given a triple $(\underline{d}, [M], [N])$ with $\underline{d} \in D(n, r)$ and a surjection $\phi : P(\underline{d}) \rightarrow M \oplus N$ we construct a corresponding pair (g_1, g_2) as follows. Recall that a surjective homomorphism $\psi : P \rightarrow M$ from a projective representation P is a projective cover if $\ker \psi \subseteq \text{rad} P$ where $\text{rad} P$ denotes the Jacobson radical of P . Decompose

$$P(\underline{d}) = f_1 \oplus f_2 \oplus c$$

such that $\phi|_{f_1}$ is a projective cover of M and $\phi|_{f_2}$ is a projective cover of N . Then the kernel $\ker \phi$ decomposes as

$$\ker \phi = f'_1 \oplus f'_2 \oplus c$$

where $f'_i = \ker \phi|_{f_i}$. Now let

$$(g_1, g_2) = (f_1 \oplus f'_2 \oplus c, f'_1 \oplus f_2 \oplus c).$$

Then $g_1 + g_2 = f_1 \oplus f_2 \oplus c$, $g_1 \cap g_2 = f'_1 \oplus f'_2 \oplus c$ and $f'_1 \subseteq \text{rad} f_1$, $f'_2 \subseteq \text{rad} f_2$, $g_1 + g_2/g_2 \simeq g_1/g_1 \cap g_2 \simeq M$ and $g_1 + g_2/g_1 \simeq g_2/g_1 \cap g_2 \simeq N$.

Given a pair of flags $(f, f') \in \mathcal{F} \times \mathcal{F}$, we have the following description of the matrix A corresponding to the orbit $e_A = [f, f']$.

Lemma 2.2. *Let $e_A = [f, f']$. Then*

- i) if $i < j$, then A_{ij} is the multiplicity of $M_{i-1, j}$ as a direct summand in $f/f \cap f'$,
- ii) if $i > j$, then A_{ij} is the multiplicity of $M_{j, i-1}$ as a direct summand in $f'/f \cap f'$, and
- iii) A_{ii} is the multiplicity of $M_{in} \rightarrow M_{in}$ as a contractible summand in the projective resolution $f \cap f' \subseteq f + f'$.

Proof. Let $(f, f') \in \mathcal{F} \times \mathcal{F}$, and $f'' = f \cap f'$ be given by $f : \{0\} = V_0 \subseteq V_1 \subseteq \dots \subseteq V_n$, $f' : \{0\} = V'_0 \subseteq V'_1 \subseteq \dots \subseteq V'_n$, and $f'' : \{0\} = V''_0 \subseteq V''_1 \subseteq \dots \subseteq V''_n$. We have

$$A_{ij} = \dim \frac{V_{i-1} + V_i \cap V'_j}{V_{i-1} + V_i \cap V'_{j-1}}.$$

Then for $i < j$ let $e_B = [f, f'']$. We have

$$\begin{aligned} B_{ij} &= \dim \frac{V_{i-1} + V_i \cap V''_j}{V_{i-1} + V_i \cap V''_{j-1}} = \dim \frac{V_{i-1} + V_i \cap V'_j}{V_{i-1} + V_i \cap V'_{j-1}} \\ &= \dim \frac{V_{i-1} + V_i \cap V_j \cap V'_j}{V_{i-1} + V_i \cap V_{j-1} \cap V'_{j-1}} = \dim \frac{V_{i-1} + V_i \cap V'_j}{V_{i-1} + V_i \cap V'_{j-1}} = A_{ij}. \end{aligned}$$

But B_{ij} is the multiplicity of $M_{i-1,j}$ as a direct summand in f/f'' , which proves *i*).

The proof of *ii*) is similar.

For the diagonal of A , we have

$$A_{ii} = \frac{V_{i-1} + V_i \cap V'_i}{V_{i-1} + V_i \cap V'_{i-1}}.$$

Write $V_i \cap V'_i = U \oplus W$ where $U \subseteq V_{i-1}$ and $W \cap V_{i-1} = 0$. Now $W = W_1 \oplus W_2$, $W_1 \subseteq V_i \cap V'_{i-1} + V_{i-1}$ and $W_2 \cap (V_i \cap V'_{i-1} + V_{i-1}) = 0$. Therefore $A_{ii} = \dim W_2$ which is the multiplicity of $M_{in} \rightarrow M_{in}$ as a contractible summand in the projective resolution $f \cap f' \rightarrow f + f'$. This proves *iii*). \square

3. THE NON-NEGATIVE q -SCHUR ALGEBRA

In this section we describe the non-negative part of a q -Schur algebra as a Hall algebra of projective resolutions of representations of the linear quiver $\Lambda = \Lambda(n)$, defined in Section 2. We also include some easy lemmas on the computation of Hall numbers for the linear quiver which are needed in subsequent sections. For deeper connections between Hall numbers and the structure constants of the q -Schur algebra, please see [14].

An orbit $[f, f'] \in \mathcal{F} \times \mathcal{F}/\text{GL}(V)$ with $f' \subseteq f$ decomposes as

$$[f, f'] = [c \oplus g, c \oplus g']$$

where $f/f' \simeq g/g'$ and $g' \subseteq \text{rad} g$. Such an orbit can be viewed as a choice of a projective resolution of g/g' obtained by adding a contractible summand c to the minimal projective resolution $0 \rightarrow g' \rightarrow g \rightarrow g/g' \rightarrow 0$.

The non-negative $\mathbb{Z}[q]$ -subalgebra $S_q^+(n, r)$ is the subalgebra of $S_q(n, r)$ with basis consisting of all orbits $[f, f']$ with $f' \subseteq f$. Similarly, the non-positive q -Schur algebra $S_q^-(n, r)$ has the corresponding basis of all orbits $[f', f]$ with $f' \subseteq f$.

Recall that the Hall number h_{MN}^L defined by Ringel [22] is the number of submodules $X \subseteq L$ satisfying $X \simeq N$ and $L/X \simeq M$.

Lemma 3.1. *Let $f_1 \supseteq f_2 \supseteq f_3$ be flags and let $e_A = [f_1, f_2]$, $e_{A'} = [f_2, f_3]$, $e_{A''} = [f'_1, f'_3]$ with $f'_1 \supseteq f'_3$, $f'_1 \simeq f_1$ and $f'_3 \simeq f_3$, $M = f_1/f_2$, $N = f_2/f_3$ and $L = f'_1/f'_3$. Then*

$$g_{A, A', A''} = h_{MN}^L.$$

Proof. Let $U = \{X \subseteq L \mid X \simeq N, L/X \simeq M\}$. We will define two mutually inverse maps between U and $S(A, A', A'')$. Given $f'_2 \in S(A, A', A'')$, we have the following commutative diagram of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & f'_3 & \longrightarrow & f'_3 & \longrightarrow & 0 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & f'_2 & \longrightarrow & f'_1 & \longrightarrow & f'_1/f'_2 \longrightarrow 0 \\ & & \downarrow & & \downarrow \pi & & \downarrow \\ 0 & \longrightarrow & f'_2/f'_3 & \longrightarrow & L & \longrightarrow & f'_1/f'_2 \longrightarrow 0, \end{array}$$

with $f_1/f_2 \simeq f'_1/f'_2 \simeq M$ and $f_2/f_3 \simeq f'_2/f'_3 \simeq N$.

Define maps $S(A, A', A'') \rightarrow U$ by $f'_2 \mapsto \pi(f'_2)$ and $U \rightarrow S(A, A', A'')$ by $X \mapsto \pi^{-1}(X)$. It is easy to check that these two maps are mutually inverse, and so the equality follows since $h_{MN}^L = |U|$. \square

Denote the (non-twisted) Ringel-Hall algebra [22] by $H_q(\Lambda)$. That is, $H_q(\Lambda)$ is the free $\mathbb{Z}[q]$ -module with basis isomorphism classes $[M]$ of representations of Λ and multiplication

$$[M][N] = \sum_L h_{MN}^L [L].$$

Mapping modules to choices of projective resolutions induces an algebra homomorphism

$$\Theta^+ : H_q(\Lambda) \rightarrow S_q^+(n, r) \text{ defined by } [M] \mapsto \sum_{\{[f, f'] \mid f' \subseteq f, f/f' \simeq M\}} [f, f']$$

with kernel spanned by those $[M]$ with the number of indecomposable direct summands bigger than r [16]. There is a similar map $\Theta^- : H_q(\Lambda) \rightarrow S_q^-(n, r)$.

As a consequence, we have the following special cases of Corollary 4.5 in [1] (see also Proposition 14.1 in [12]). The assumptions are as in Lemma 3.1.

Corollary 3.2. *We have*

$$g_{A+D, A'+D, A''+D} = h_{M, N}^L,$$

for any diagonal matrix $D = \text{diag}(d_1, \dots, d_n)$ with $d_i \geq 0$.

This yields a different proof of Theorem 14.27 in [12], which we restate here as follows.

Theorem 3.3. *Hall algebra $H_q(\Lambda)$ is isomorphic to the algebra with basis all formal sums*

$$\left\{ \sum_{f/g \cong M} [f, g] \mid M \in \text{mod } kQ \right\},$$

over all decompositions of r into n parts and for all $r \geq 1$, and the multiplication as in $S_q(n, r)$.

As before, let M_{ij} be the indecomposable representation of Λ supported on the interval $[i, j]$. Let $M_{ij} \leq M_{i'j'}$ if $j < j'$ or $i \leq i'$ if $j = j'$. This is a total order on the indecomposable representations of Λ . Observe that if $M_{ij} \leq M_{i'j'}$ then $\text{Ext}^1(M_{i'j'}, M_{ij}) = 0$, but that the converse is not true in general. We state a lemma which follows easily from Lemma 3.1 and the corresponding computations in the Ringel-Hall algebra.

Lemma 3.4. *Suppose flags $f \supseteq g$ with $f/g = \oplus_{ij} M_{ij}^{m_{ij}}$. Then there exists a filtration $f = f_n \supset f_{n-1} \supset \dots \supset f_1 = g \supset 0$ with indecomposable factors $f_i/f_{i-1} \leq f_{i+1}/f_i$ for all i , such that*

$$[f_n, f_{n-1}] \cdots [f_2, f_1] = [f_n, f_1] \prod [m_{ij}]!$$

and m_{ij} is the multiplicity of M_{ij} as a subfactor in the filtration.

4. BASES OF $S_q(n, r)$

In this section we describe a basis for $S_q(n, r)$ using the non-negative and non-positive subalgebras defined in the previous section. Let

$$\mathcal{B} = \{[f_1, f_1 + f_2][f_1 + f_2, f_2] \mid [f_1, f_2] \in \mathcal{F} \times \mathcal{F}/\text{GL}(V)\}.$$

Lemma 2.1 shows that the map $\mathcal{F} \times \mathcal{F}/\text{GL}(V) \rightarrow \mathcal{B}$ given by

$$[f_1, f_2] \mapsto [f_1, f_1 + f_2][f_1 + f_2, f_2]$$

is a well defined surjection of sets. We shall see that \mathcal{B} is a $\mathbb{Z}[q]$ -basis of $S_q(n, r)$. Note that there is a similar basis \mathcal{B}' of $S_q(n, r)$ consisting of elements of the form $[f_1, f_1 \cap f_2][f_1 \cap f_2, f_2]$.

Lemma 4.1. *Let $(f_1, f_2) \in \mathcal{F} \times \mathcal{F}$ and let $e_A = [f_1, f_1 + f_2]$ and $e_{A'} = [f_1 + f_2, f_2]$. Then*

$$[f_1, f_1 + f_2][f_1 + f_2, f_2] = [f_1, f_2] + \sum_{\{e_{A''} = [f'_1, f'_2] \mid f'_1 + f'_2 \subsetneq f_1 + f_2\}} g_{A, A', A''} [f'_1, f'_2].$$

Proof. Suppose that $[f'_1, f'_2]$ is one of the terms with a non-zero coefficient in the sum

$$[f_1, f_1 + f_2][f_1 + f_2, f_2] = \sum g_{A, A', A''} [f'_1, f'_2].$$

Then there exists an $f \in \mathcal{F}$ such that $(f'_1, f) \simeq (f_1, f_1 + f_2)$ and $(f, f'_2) \simeq (f_1 + f_2, f_2)$. Thus $f'_1, f'_2 \subseteq f$ and so $f'_1 + f'_2 \subseteq f$. Note that if $f'_1 + f'_2 = f$, then $(f'_1, f) = (f'_1, f'_1 + f'_2) \simeq (f_1, f_1 + f_2)$ and $(f, f'_2) = (f'_1 + f'_2, f'_2) \simeq (f_1 + f_2, f_2)$. Therefore $(f'_1 + f'_2)/f'_i \simeq (f_1 + f_2)/f_i$ for $i = 1, 2$. By Lemma 2.1, $(f'_1, f'_2) \simeq (f_1, f_2)$. Moreover $g_{A, A', A''} = 1$ for $e_A = [f_1, f_2]$. The lemma follows. \square

There is a similar formula for the product $[f_1, f_1 \cap f_2][f_1 \cap f_2, f_2]$.

Theorem 4.2. *The set \mathcal{B} is a $\mathbb{Z}[q]$ -basis of $S_q(n, r)$.*

Proof. By induction on the size of $f_1 + f_2$ and Lemma 4.1, any basis element $[f_1, f_2]$ can be written as a $\mathbb{Z}[q]$ -linear combination of elements in \mathcal{B} . This proves that \mathcal{B} spans $S_q(n, r)$ as a $\mathbb{Z}[q]$ -module. Then \mathcal{B} is in bijection with the basis $\mathcal{F} \times \mathcal{F}/\text{GL}(V)$, which proves that \mathcal{B} is a basis of $S_q(n, r)$. \square

Each basis element in \mathcal{B} has a decomposition

$$[c \oplus f'_1 \oplus f_2, c \oplus f_1 \oplus f_2][c \oplus f_1 \oplus f_2, c \oplus f_1 \oplus f'_2]$$

where $f'_1 \subseteq \text{rad } f_1$ and $f'_2 \subseteq \text{rad } f_2$.

Lemma 4.3. *Let $f_1 \subseteq g \supseteq f_2$ be flags in \mathcal{F} . Then $(f_1, g) \simeq (h_1, h_1 + h_2)$ and $(g, f_2) \simeq (h_1 + h_2, h_2)$ for a pair of flags $h_1, h_2 \in \mathcal{F}$, if and only if there is a surjective map $g \rightarrow g/f_1 \oplus g/f_2$.*

Proof. Assume that $\phi : g \rightarrow g/f_1 \oplus g/f_2$ is surjective. Let $h_1 = \phi^{-1}(g/f_2)$ and $h_2 = \phi^{-1}(g/f_1)$. Then $(h_1, g) \simeq (f_1, g)$ and $(g, h_2) \simeq (g, f_2)$ and $g = \phi^{-1}(g/f_1 \oplus g/f_2) = h_1 + h_2$.

The converse follows since the map $\pi : h_1 + h_2 \rightarrow (h_1 + h_2)/h_1 \oplus (h_1 + h_2)/h_2$ is surjective. \square

In particular the lemma shows that a surjective map $g \rightarrow g/f_1 \oplus g/f_2$ implies $[f_1, g][g, f_2] \in \mathcal{B}$. The following example shows that the converse is not true.

Example 4.4. *Let $n = 2, r = 2, V = \text{span}\{x_1, x_2\}$, $f_i : 0 \subseteq kx_i$ for $i = 1, 2$, and $g : V \subseteq V$. Then*

$$[f_1, g][g, f_2] = [f_1, f_2] + [f_1, f_1] = [f_1, f_1 + f_2][f_1 + f_2, f_2] \in \mathcal{B},$$

with no surjective map $g \rightarrow g/f_1 \oplus g/f_2$. In this case $g \not\supseteq f_1 + f_2$.

5. QUIVER AND RELATIONS FOR q -SCHUR ALGEBRAS

In this section we present an algebra using quivers and binomial relations, which will be shown to be the 0-Schur algebra in Section 7. This will lead to a presentation of the q -Schur over a base ring where q is not invertible. Also, following from the relations, the 0-Schur algebra has a multiplicative basis of paths which will be constructed geometrically in Section 6 and 7.

As before, let $D(n, r)$ be the set of decompositions of r into n parts. For $\underline{d} \in D(n, r)$, we define $\underline{d} + \alpha_i$ by

$$(\underline{d} + \alpha_i)_j = \begin{cases} d_i + 1, & \text{if } i = j, \\ d_j, & \text{if } i \neq j. \end{cases}$$

Let $\Sigma(n, r)$ be the quiver with vertices $K_{\underline{d}}$ and arrows $E_{i, \underline{d}}$ and $F_{i, \underline{d}}$,

$$\begin{array}{ccc} & \xrightarrow{F_{i, \underline{d} + \alpha_i - \alpha_{i+1}}} & \\ K_{\underline{d} + \alpha_i - \alpha_{i+1}} & & K_{\underline{d}} \\ & \xleftarrow{E_{i, \underline{d}}} & \end{array}$$

where $\underline{d}, \underline{d} + \alpha_i - \alpha_{i+1} \in D(n, r)$. The vertices can be drawn on a simplex, where the vertices $K_{\underline{d}}$ with $d_i = 0$ for some $i = 0$ are on the boundary, and vertices $K_{\underline{d}}$ with $d_i \neq 0$ for all i are in the interior of the simplex.

To simplify our formulas we define

$$\begin{aligned} E_{i, \underline{d}} &= 0 \text{ if } \begin{cases} \underline{d} \notin D(n, r), \\ i = n, \text{ or } \\ d_{i+1} = 0 \end{cases}, \\ F_{i, \underline{d}} &= 0 \text{ if } \begin{cases} \underline{d} \notin D(n, r), \\ i = n, \text{ or } \\ d_i = 0 \end{cases}, \\ K_{\underline{d}} &= 0 \text{ if } \underline{d} \notin D(n, r), \\ E_i &= \sum_{\underline{d}} E_{i, \underline{d}} \text{ and } F_i = \sum_{\underline{d}} F_{i, \underline{d}}. \end{aligned}$$

For a commutative ring R , denote by $R\Sigma(n, r)$ the path R -algebra of $\Sigma(n, r)$, which is the free R -module with basis all paths in $\Sigma(n, r)$, and multiplication given by composition of paths. The vertices $K_{\underline{d}}$ form an orthogonal set of idempotents in $R\Sigma(n, r)$ and the composition of two paths p and q is pq , if q ends where p starts, and zero otherwise. Recall that a relation in $R\Sigma(n, r)$ is a R -linear combination of paths with common starting and ending vertex

$$\rho = \sum_i r_i p_i, r_i \in R, p_i \text{ a path.}$$

Let $I(n, r) \subseteq \mathbb{Z}[q]\Sigma(n, r)$ be the ideal generated by the relations

$$\begin{aligned} P_{ij, \underline{d}} &= K_{\underline{d} + p_{ij}} P_{ij} K_{\underline{d}}, \\ N_{ij, \underline{d}} &= K_{\underline{d} - p_{ij}} N_{ij} K_{\underline{d}}, \text{ and} \\ C_{ij, \underline{d}} &= K_{\underline{d} + \alpha_i + \alpha_{j+1} - \alpha_{i+1} - \alpha_j} C_{ij} K_{\underline{d}}, \end{aligned}$$

where

$$\begin{aligned} p_{ij} &= \begin{cases} 2\alpha_i + \alpha_j - 2\alpha_{i+1} - \alpha_{j+1}, & \text{if } |i-j| = 1, \\ \alpha_i + \alpha_j - \alpha_{i+1} - \alpha_{j+1}, & \text{if } |i-j| > 1; \end{cases} \\ P_{ij} &= \begin{cases} E_i^2 E_j - (q+1)E_i E_j E_i + qE_j E_i^2 & \text{for } i = j-1, \\ qE_i^2 E_j - (q+1)E_i E_j E_i + E_j E_i^2 & \text{for } i = j+1, \\ E_i E_j - E_j E_i, & \text{otherwise;} \end{cases} \\ N_{ij} &= \begin{cases} qF_i^2 F_j - (q+1)F_i F_j F_i + F_j F_i^2 & \text{for } i = j-1, \\ F_i^2 F_j - (q+1)F_i F_j F_i + qF_j F_i^2 & \text{for } i = j+1, \\ F_i F_j - F_j F_i, & \text{otherwise;} \end{cases} \end{aligned}$$

and

$$C_{ij} = E_i F_j - F_j E_i - \delta_{ij} \sum_{\underline{d}} \frac{q^{d_i} - q^{d_{i+1}}}{q-1} K_{\underline{d}}.$$

Let

$$e_{i,\underline{d}} = [f, f'], f_{i,\underline{d}+\alpha_i-\alpha_{i+1}} = [f', f] \text{ and } k_{\underline{d}} = [h, h]$$

where $(f, f') \in \mathcal{F} \times \mathcal{F}$ with $f' \subseteq f$, $f/f' \simeq S_i$, and $f', h \in \mathcal{F}_{\underline{d}}$.

Lemma 5.1. *There is a homomorphism of $\mathbb{Z}[q]$ -algebras*

$$\phi : \mathbb{Z}[q]\Sigma(n, r)/I(n, r) \rightarrow S_q(n, r)$$

defined by

$$\phi(E_{i,\underline{d}}) = e_{i,\underline{d}}, \phi(F_{i,\underline{d}}) = f_{i,\underline{d}} \text{ and } \phi(K_{\underline{d}}) = k_{\underline{d}}.$$

Proof. By Lemma 1.1, the relations P_{ij} , N_{ij} and C_{ij} hold in $S_q(n, r)$, and so ϕ is an algebra homomorphism. \square

We remark that the relations P_{ij} and N_{ij} hold in $S_q(n, r)$ also follows from Lemma 3.1 and the proposition in Section 2 of [23], and that the lemma can be deduced from Lemma 5.6 in [1].

The homomorphism ϕ not surjective in general, and so this is not a presentation of the q -Schur algebra over $\mathbb{Z}[q]$, since for instance $[m]$ is not invertible in $\mathbb{Z}[q]$.

5.1. Change of rings. We need the following change of rings lemma for presentations of algebras using quivers with relations. The proof is similar to an argument at the end of Chapter 5 in [17]. Let $\psi : R \rightarrow S$ be a homomorphism of commutative rings, which gives S the structure of an R -algebra. Let Σ be a quiver, and let $I \subseteq R\Sigma$ be an ideal. There are induced map of R -algebras $\psi : R\Sigma \rightarrow S\Sigma$ and $R\Sigma/I \rightarrow S\Sigma/S\psi(I)$, where $S\psi(I) \subseteq S\Sigma$ is the ideal generated by $\psi(I)$.

Lemma 5.2. *The induced map $(R\Sigma/I) \otimes_R S \rightarrow S\Sigma/S\psi(I)$ is an isomorphism of R -algebras.*

Proof. The natural isomorphism $R \otimes_R S \rightarrow S$ of R -algebras induces an R -algebra isomorphism

$$m : R\Sigma \otimes_R S \rightarrow S\Sigma.$$

Applying the functor $- \otimes_R S$ to the short exact sequence

$$0 \rightarrow I \xrightarrow{i} R\Sigma \rightarrow R\Sigma/I \rightarrow 0$$

gives us the exact sequence

$$I \otimes_R S \xrightarrow{j} S\Sigma \rightarrow (R\Sigma/I) \otimes_R S \rightarrow 0$$

where $j = m \circ (i \otimes S)$, which shows that

$$(R\Sigma/I) \otimes_R S \simeq S\Sigma/\text{im}(m \circ (i \otimes S)).$$

As $\text{im}(m \circ (i \otimes S)) = S\psi(I)$, the proof is complete. \square

5.2. **Coefficients in $\mathbb{Q}(v)$.** Let $v = \sqrt{q}$ and

$$S_v(n, r) = S_q(n, r) \otimes_{\mathbb{Z}[q]} \mathbb{Q}(v).$$

Lemma 5.3. *There is an isomorphism of $\mathbb{Q}(v)$ -algebras $\mathbb{Q}(v)\Sigma(n, r)/\mathbb{Q}(v)I(n, r) \rightarrow S_v(n, r)$, where $E_{i, \underline{d}} \mapsto e_{i, \underline{d}}$, $F_{i, \underline{d}} \mapsto f_{i, \underline{d}}$ and $K_{\underline{d}} \mapsto k_{\underline{d}}$.*

Proof. Let $\tilde{E}_i = \sum_{\underline{d}} v^{-d_i+1} e_{i, \underline{d}}$, $\tilde{F}_i = \sum_{\underline{d}} v^{-d_i+1} f_{i, \underline{d}}$ and $\tilde{K}_{\underline{d}} = k_{\underline{d}}$, and by abuse of notation, in this proof we let $E_i = \sum_{\underline{d}} e_{i, \underline{d}}$, $F_i = \sum_{\underline{d}} f_{i, \underline{d}}$ and $K_{\underline{d}} = k_{\underline{d}}$. Then both $\{\tilde{E}_i, \tilde{F}_j, \tilde{K}_{\underline{d}}\}$ and $\{E_i, F_j, K_{\underline{d}}\}$ generate $S_v(n, r)$. Moreover, by a straightforward computation, $\tilde{E}_i, \tilde{F}_j, \tilde{K}_{\underline{d}}$ satisfy the defining relations in Theorem 4' in [10] if and only if $E_i, F_j, K_{\underline{d}}$ satisfy the relations P_{ij}, N_{ij} and C_{ij} . Therefore we have the isomorphism as required. \square

Proposition 5.4. *The induced map $\phi \otimes \mathbb{Q}(v) : \mathbb{Z}[q]\Sigma(n, r)/I(n, r) \otimes_{\mathbb{Z}[q]} \mathbb{Q}(v) \rightarrow S_q(n, r) \otimes_{\mathbb{Z}[q]} \mathbb{Q}(v)$ is a $\mathbb{Q}(v)$ -algebra isomorphism, where ϕ is as in Lemma 5.1.*

Proof. By Lemma 5.2, the natural inclusion $\psi : \mathbb{Z}[q] \rightarrow \mathbb{Q}(v)$, induces an isomorphism

$$\mathbb{Z}[q]\Sigma(n, r)/I(n, r) \otimes_{\mathbb{Z}[q]} \mathbb{Q}(v) \simeq \mathbb{Q}(v)\Sigma(n, r)/\mathbb{Q}(v)I(n, r),$$

which composed with the isomorphism in Lemma 5.3 is $\phi \otimes \mathbb{Q}(v)$. Thus the proposition follows. \square

Since q is invertible in $S_v(n, r)$, we cannot obtain the 0-Schur algebra by specialising $q = 0$.

5.3. **A presentation of q -Schur algebra over \mathcal{Q} .** Now we choose an intermediate ring $\mathbb{Z}[q] \subseteq \mathcal{Q} \subseteq \mathbb{Q}(v)$ such that q is non-invertible in \mathcal{Q} , and we will prove in Section 7 that $\phi \otimes \mathcal{Q}$ is an isomorphism.

Let \mathcal{Q} be obtained from $\mathbb{Z}[q]$ by inverting all polynomials of the form $1 + qf(q)$. In particular, all $[m]$ for $m \in \mathbb{N}$ are inverted. Clearly, $\mathbb{Z}[q] \subseteq \mathcal{Q} \subseteq \mathbb{Q}(q)$. Note that q is not invertible in \mathcal{Q} and so the specialisation $q = 0$ is possible.

Proposition 5.5. *The induced map $\phi \otimes \mathcal{Q} : \mathbb{Z}[q]\Sigma(n, r)/I(n, r) \otimes_{\mathbb{Z}[q]} \mathcal{Q} \rightarrow S_q(n, r) \otimes_{\mathbb{Z}[q]} \mathcal{Q}$ is a surjective \mathcal{Q} -algebra homomorphism.*

Proof. The image of $\phi \otimes \mathcal{Q}$ is the subalgebra of $S_q(n, r) \otimes_{\mathbb{Z}[q]} \mathcal{Q}$ generated by the set of all $e_{i, \underline{d}}, f_{i, \underline{d}}$ and $k_{\underline{d}}$. Lemma 3.4 shows that the $\mathbb{Z}[q]$ -subalgebra of $S_q^+(n, r)$ generated by all $e_{i, \underline{d}}$ and $k_{\underline{d}}$ contains all

$$[f, g] \prod [m_{ij}]!$$

where $g \subseteq f$ and m_{ij} is the multiplicity of M_{ij} as a direct summand in f/g . Since $[m]$ is invertible in \mathcal{Q} for any m , the image contains $S_q^+(n, r) \otimes_{\mathbb{Z}[q]} \mathcal{Q}$. Similarly, the image contains $S_q^-(n, r) \otimes_{\mathbb{Z}[q]} \mathcal{Q}$. By Theorem 4.2 the map is surjective. \square

By Lemma 5.2, $\mathcal{Q}\Sigma(n, r)/\mathcal{Q}I(n, r) \simeq \mathbb{Z}[q]\Sigma(n, r)/I(n, r) \otimes_{\mathbb{Z}[q]} \mathcal{Q}$. We have the following theorem, which will be proven in Section 7.

Theorem 5.6. *The induced map $\phi \otimes \mathcal{Q} : \mathbb{Z}[q]\Sigma(n, r)/I(n, r) \otimes_{\mathbb{Z}[q]} \mathcal{Q} \rightarrow S_q(n, r) \otimes_{\mathbb{Z}[q]} \mathcal{Q}$ is a \mathcal{Q} -algebra isomorphism.*

6. A GENERIC ALGEBRA

In this section let k be algebraically closed. We define a generic multiplication of orbits in $\mathcal{F} \times \mathcal{F}$ and obtain a \mathbb{Z} -algebra $G(n, r)$, which we call a generic algebra. This multiplication generalises the one for positive 0-Schur algebras in [24] and is similar to the product defined by Reineke [21] for Hall algebras. We also give generators for $G(n, r)$.

Let $\Delta : \mathcal{F} \times \mathcal{F} \times \mathcal{F} \rightarrow (\mathcal{F} \times \mathcal{F}) \times (\mathcal{F} \times \mathcal{F})$ be the morphism given by

$$\Delta(p_1, p_2, p_3) = ((p_1, p_2), (p_2, p_3)).$$

Let

$$\pi : \mathcal{F} \times \mathcal{F} \times \mathcal{F} \rightarrow \mathcal{F} \times \mathcal{F}$$

be the projection onto the left and right component. The map π is open, and Δ is a closed embedding.

Given two orbits e_A and $e_{A'}$, define

$$S(A, A') = \pi \Delta^{-1}(e_A \times e_{A'})$$

That is, $S(A, A')$ is the union of the orbits with non-zero coefficient in the product $e_A \cdot e_{A'}$ in $S_q(n, r)$.

Lemma 6.1. *The closure of $S(A, A')$ in $\mathcal{F} \times \mathcal{F}$ is irreducible.*

Proof. Let $[f_1, f_2] = e_A$, $[f_3, f_4] = e_{A'}$, and $S = \Delta^{-1}(e_A \times e_{A'})$. We first show that S is irreducible. If $f_2 \not\simeq f_3$ then S is empty, and we are done. So we may assume that $f_2 = f_3$. Let $(p_1, p_2, p_3) \in S$ then there exists $g \in \text{GL}(V)$ such that $(p_2, p_3) = g(f_3, f_4)$ and $g(g^{-1}p_1, f_3, f_4) = (p_1, p_2, p_3)$, where $\text{GL}(V)$ acts diagonally. Since $(g^{-1}p_1, f_3) \simeq (f_1, f_3)$, there is an $a \in \text{Aut}f_3$ such that $g^{-1}p_1 = af_1$. Hence S is the image of the morphism

$$\text{Aut}f_3 \times \text{GL}(V) \rightarrow \mathcal{F} \times \mathcal{F} \times \mathcal{F}$$

given by

$$(a, g) \mapsto (gaf_1, g f_3, g f_4)$$

and is therefore irreducible. Now $S(A, A') = \pi(S)$, and so its closure is irreducible. \square

Since there are only finitely many orbits in $S(A, A')$, as a consequence of Lemma 6.1, we have the following corollary.

Corollary 6.2. *There is a unique open $\text{GL}(V)$ -orbit in $S(A, A')$.*

We define a new multiplication

$$e_A \star e_{A'} = e_{A''},$$

if $S(A, A')$ is non-empty and $e_{A''}$ is the open orbit in $S(A, A')$, and

$$e_A \star e_{A'} = 0$$

if $S(A, A')$ is empty.

Proposition 6.3. *The free \mathbb{Z} -module $G(n, r)$ with the product \star is a \mathbb{Z} -algebra.*

Proof. We need only to show that \star is associative, that is, for any $[f_1, f_2], [f_3, f_4], [f_5, f_6] \in \mathcal{F} \times \mathcal{F}/\text{GL}(V)$,

$$([f_1, f_2] \star [f_3, f_4]) \star [f_5, f_6] = [f_1, f_2] \star ([f_3, f_4] \star [f_5, f_6])$$

Following the definition, we see that if one side of the equality is zero, then so is the other side. We now suppose that both sides are not zero, that is, $f_2 \simeq f_3$ and $f_4 \simeq f_5$. By change of basis we may assume that $f_2 = f_3$ and $f_4 = f_5$. Let

$$\begin{aligned} T_1 &= \{(p_1, p_2, p_3, p_4) \mid (p_1, p_2) \simeq (f_1, f_2), (p_2, p_3) \simeq (f_3, f_4), (p_3, p_4) \simeq (f_5, f_6)\}, \\ T_2 &= \{(p_1, p_3, p_4) \mid \exists p \text{ such that } (p_1, p) \simeq (f_1, f_2), (p, p_3) \simeq (f_3, f_4), (p_3, p_4) \simeq (f_5, f_6)\}, \\ T_3 &= \{(p_1, p_2, p_4) \mid \exists p \text{ such that } (p_1, p_2) \simeq (f_1, f_2), (p_2, p) \simeq (f_3, f_4), (p, p_4) \simeq (f_5, f_6)\}, \\ T_4 &= \{(p_1, p_4) \mid \exists p, p' \text{ such that } (p_1, p) \simeq (f_1, f_2), (p, p') \simeq (f_3, f_4), (p', p_4) \simeq (f_5, f_6)\}. \end{aligned}$$

We have natural surjections

$$\pi_{ij} : T_i \rightarrow T_j$$

for $(i, j) = (1, 2), (1, 3), (2, 4), (3, 4)$. Similar to the proof of Lemma 6.1, we see that T_1 is irreducible, and so the closures of all the T_i are irreducible. In particular, there is a unique open orbit \mathcal{O} in T_4 . Then $\pi_{24}^{-1}(\mathcal{O})$ intersects with the open subset of T_2 , consisting of triples (p_1, p_3, p_4) with $[p_1, p_3]$ open in $S(A, A')$. That is, $([f_1, f_2] \star [f_3, f_4]) \star [f_5, f_6]$ is the open orbit \mathcal{O} in T_4 . Similarly, $[f_1, f_2] \star ([f_3, f_4] \star [f_5, f_6])$ is also the open orbit \mathcal{O} . Therefore the equality holds and so \star is associative. \square

The following is a direct consequence of the definition of the product in $G(n, r)$.

Corollary 6.4. *The set $\mathcal{F} \times \mathcal{F}/\text{GL}(V)$ is a multiplicative basis of $G(n, r)$.*

In addition to the basis of $G(n, r)$ consisting of orbits $[f_1, f_2]$ we can also consider bases analogous to the bases \mathcal{B} and \mathcal{B}' defined in Section 4 for the q -Schur algebra. We conclude this section by showing that these three bases of $G(n, r)$ coincide.

Lemma 6.5. *Let $(f_1, f_2) \in \mathcal{F} \times \mathcal{F}$. Then $[f_1, f_1 + f_2] \star [f_1 + f_2, f_2] = [f_1, f_2] = [f_1, f_1 \cap f_2] \star [f_1 \cap f_2, f_2]$.*

Proof. We prove the first equality. Let $e_A = [f_1, f_1 + f_2]$ and $e_{A'} = [f_1 + f_2, f_2]$. We prove that the orbit $[f_1, f_2]$ is open in $S(A, A')$. For any $(f'_1, f'_2) \in S(A, A')$, $f'_1 + f'_2$ is isomorphic to a subflag of $f_1 + f_2$. By Lemma 4.1, for $(f'_1, f'_2) \in S(A, A')$, we have $(f'_1, f'_2) \simeq (f_1, f_2)$ if and only if $f'_1 + f'_2 \simeq f_1 + f_2$. That the dimension of $f'_1 + f'_2$ is maximal is an open condition, and therefore $e_A \star e_{A'} = [f_1, f_2]$.

Similarly, $[f_1, f_1 \cap f_2] \star [f_1 \cap f_2, f_2] = [f_1, f_2]$. \square

We now prove that the \mathbb{Z} -algebra $G(n, r)$ is generated by the orbits $e_{i, \underline{d}}$, $f_{i, \underline{d}}$ and $k_{\underline{d}}$. Recall that a representation X is said to be a generic extension of N by M , if the stabiliser of X is minimal among all representations that are extensions of N by M .

Lemma 6.6. [24] *Let $f \supseteq g \supseteq h$ be flags. Then $[f, h] = [f, g] \star [g, h]$ if and only if f/h is a generic extension of f/g by g/h .*

For an interval $[i, j]$ in $\{1, \dots, n\}$ and $\underline{d} \in D(n, r)$ with $\underline{d} - \alpha_{j+1}$ non-negative, let

$$e(i, j, \underline{d}) = e_{i, \underline{d} + \alpha_{i+1} - \alpha_{j+1}} \star \dots \star e_{j, \underline{d}}.$$

Similaly, let $f(i, j, \underline{d}) = f_{j, \underline{d} - \alpha_{i+1} + \alpha_{j+1}} \star \dots \star f_{i, \underline{d}}$ for $\underline{d} - \alpha_{i+1}$ non-negative.

Lemma 6.7. *Let $f \supseteq h$ be flags with $h \in \mathcal{F}_{\underline{d}}$ and $f/h \simeq M_{ij}$. Then $[f, h] = e(i, j, \underline{d})$ and $[h, f] = f(i, j, \underline{d} + \alpha_i - \alpha_{j+1})$.*

Proof. If $i = j$, then $[f, h] = e_{i, \underline{d}}$. Now assume $j > i$. Then there is $f \supseteq g \supseteq h$ with $f/g \simeq M_{i, j-1}$ and $g/h \simeq M_{jj}$. Since f/h is a generic extension of f/g by the simple g/h , the lemma follows from Lemma 6.6 by induction. \square

Using the order \leq on representations defined in Section 3, we can write each orbit $[f, g]$ with $f \supseteq g$ as a product over indecomposable summands of f/g .

Lemma 6.8. *Let $f \supseteq g$ be flags with $f/g \simeq \bigoplus_{i=1}^t M_i$ and $M_i \leq M_{i+1}$. Then there is a filtration $f = f_t \supset f_{t-1} \supset \dots \supset f_0 = g \supset 0$ with indecomposable factors $M_i = f_i/f_{i-1}$ and $[f, g] = [f_t, f_{t-1}] \star \dots \star [f_1, f_0]$.*

Proof. The lemma follows from the vanishing of extension groups along the filtration and Lemma 6.6. \square

Lemma 6.9. *The \mathbb{Z} -algebra $G(n, r)$ is generated by the orbits $e_{i, \underline{d}}$, $f_{i, \underline{d}}$ and $k_{\underline{d}}$.*

Proof. Lemma 6.7 and Lemma 6.8 imply that any orbit $[f, g]$ with $f \supseteq g$ is in the subalgebra of $G(n, r)$ generated by $e_{i, \underline{d}}$ and $k_{\underline{d}}$. The lemma now follows from Lemma 6.5. \square

Following Lemma 6.5, 6.7 and 6.8, we obtain the following basis of $G(n, r)$ in terms the generators $e_{i, \underline{d}}$ and $f_{i, \underline{d}}$.

Lemma 6.10. *The \mathbb{Z} -algebra $G(n, r)$ has a basis consisting of all $k_{\underline{d}}$ and all non-zero monomials*

$$e(i_s, j_s, \underline{d}_s) \star \dots \star e(i_1, j_1, \underline{d}_1) \star f(i'_1, j'_1, \underline{d}'_1) \star \dots \star f(i'_t, j'_t, \underline{d}'_t),$$

where $M_{i_l j_l} \leq M_{i_{l+1} j_{l+1}}$, $M_{i'_l j'_l} \leq M_{i'_{l+1} j'_{l+1}}$ and $\underline{d}_1 \geq \sum_l \alpha_{j_l+1} + \sum_l \alpha_{j'_l+1}$.

Proof. We need only show that for any such monomial

$$e(i_s, j_s, \underline{d}_s) \star \dots \star e(i_1, j_1, \underline{d}_1) \star f(i'_1, j'_1, \underline{d}'_1) \star \dots \star f(i'_t, j'_t, \underline{d}'_t)$$

there is an orbit $[f, g]$ such that

$$[f, f \cap g] = e(i_s, j_s, \underline{d}_s) \star \dots \star e(i_1, j_1, \underline{d}_1) \text{ and } [f \cap g, g] = f(i'_1, j'_1, \underline{d}'_1) \star \dots \star f(i'_t, j'_t, \underline{d}'_t).$$

Write $\underline{d} = \underline{c} + \underline{d}' + \underline{d}''$, where $\underline{d}' = \sum_l \alpha_{j_l+1}$ and $\underline{d}'' = \sum_l \alpha_{j'_l+1}$. Consider $P(d)$ as a flag in V , and decompose as $P(d) = P(c) \oplus P(d') \oplus P(d'')$. Let $Q(d')$ and $Q(d'')$ be flags containing $P(d')$ and $P(d'')$, respectively, such that

$$Q(d')/P(d') \simeq \bigoplus M_{i_l j_l} \text{ and } Q(d'')/P(d'') \simeq \bigoplus M_{i'_l j'_l}.$$

Let $f = P(c) \oplus Q(d') \oplus P(d'')$ and $g = P(c) \oplus P(d') \oplus Q(d'')$, then $[f, g]$ is an orbit as required. \square

We compute the multiplication in $G(n, r)$ of an arbitrary element with a generator.

Lemma 6.11. *Let $e_A \subseteq \mathcal{F}_{\underline{d}} \times \mathcal{F}$.*

- i) *If $d_{i+1} > 0$, then $e_{i, \underline{d}} \star e_A = e_X$ where $X = A + E_{i, p} - E_{i+1, p}$ and $p = \max\{j \mid A_{i+1, j} > 0\}$.*
- ii) *If $d_i > 0$, then $f_{i, \underline{d}} \star e_A = e_Y$ where $Y = A - E_{i, p} + E_{i+1, p}$ and $p = \min\{j \mid A_{i, j} > 0\}$.*

Proof. We prove i). By Lemma 1.1, the orbit e_X has a non-zero coefficient in the product $e_{i, \underline{d}} \cdot e_A$ in $S_q(n, r)$. Now, by Lemma 2.2 in [1], among all terms $A + E_{i, j} - E_{i+1, j}$ with $A_{i+1, j} > 0$, the elements in the orbit e_X has the smallest stabiliser, and so $e_{i, \underline{d}} \star e_A = e_X$.

The proof of ii) is similar. \square

7. A GEOMETRIC REALISATION OF THE 0-SCHUR ALGEBRA

In this section we first give a presentation of $G(n, r)$ using quivers and relations. Then we show that $S_0(n, r)$ and $G(n, r)$ are isomorphic as \mathbb{Z} -algebras by an isomorphism which is the identity on the closed orbits $e_{i,\underline{d}}$, $f_{i,\underline{d}}$ and $k_{\underline{d}}$. Finally, we prove Theorem 5.6.

7.1. A presentation of $G(n, r)$. Let $\Sigma(n, r)$, E_i and F_i be as in Section 5. Let

$$P_{ij}(0) = \begin{cases} E_i^2 E_j - E_i E_j E_i & \text{for } i = j - 1, \\ -E_i E_j E_i + E_j E_i^2 & \text{for } i = j + 1, \\ E_i E_j - E_j E_i, & \text{otherwise;} \end{cases}$$

$$N_{ij}(0) = \begin{cases} -F_i F_j F_i + F_j F_i^2 & \text{for } i = j - 1, \\ F_i^2 F_j - F_i F_j F_i & \text{for } i = j + 1, \\ F_i F_j - F_j F_i, & \text{otherwise;} \end{cases}$$

and

$$C_{ij}(0) = E_i F_j - F_i E_j - \delta_{ij} \sum_{\underline{d}} \lambda_{ij}(\underline{d}) \cdot K_{\underline{d}},$$

where

$$\lambda_{ij}(\underline{d}) = \begin{cases} 1 & \text{if } d_i > d_{i+1} = 0, \\ -1 & \text{if } d_{i+1} > d_i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

That is, $P_{ij}(0)$, $N_{ij}(0)$ and $C_{ij}(0)$ are obtained by evaluating P_{ij} , N_{ij} and C_{ij} at $q = 0$.

Let $I_0(n, r) \subseteq \mathbb{Z}\Sigma(n, r)$ be the ideal generated by $P_{ij,d}(0)$, $N_{ij,d}(0)$, and $C_{ij,d}(0)$.

Lemma 7.1. $\mathbb{Z}\Sigma(n, r)/I_0(n, r)$ has a multiplicative basis of paths in $\Sigma(n, r)$.

Proof. The lemma holds since each relation $P_{ij,\underline{d}}(0)$, $N_{ij,\underline{d}}(0)$, and $C_{ij,\underline{d}}(0)$ is a binomial in $E_{i,\underline{d}}$, $F_{i,\underline{d}}$ and $K_{\underline{d}}$. This is obvious for $P_{ij,\underline{d}}(0)$, $N_{ij,\underline{d}}(0)$. For $C_{ij,\underline{d}}(0)$, if the coefficient of $K_{\underline{d}}$ is nonzero then either $C_{ij,\underline{d}}(0) = K_{\underline{d}} E_i F_j K_{\underline{d}} - K_{\underline{d}}$ or $C_{ij,\underline{d}}(0) = K_{\underline{d}} F_i E_j K_{\underline{d}} - K_{\underline{d}}$. \square

For an interval $[i, j]$ in $\{1, \dots, n\}$ and $\underline{d} \in D(n, r)$ with $\underline{d} - \alpha_{j+1}$ non-negative, let

$$E(i, j, \underline{d}) = E_{i,\underline{d}+\alpha_{i+1}-\alpha_{j+1}} \cdots E_{j,\underline{d}}$$

and $F(i, j, \underline{d}) = F_{j,\underline{d}-\alpha_{i+1}+\alpha_{j+1}} \cdots F_{i,\underline{d}}$ for $\underline{d} - \alpha_{i+1}$ non-negative.

Theorem 7.2. The map $\eta : \mathbb{Z}\Sigma(n, r)/I_0(n, r) \rightarrow G(n, r)$ given by $\eta(E_{i,\underline{d}}) = e_{i,\underline{d}}$, $\eta(F_{i,\underline{d}}) = f_{i,\underline{d}}$ and $\eta(K_{\underline{d}}) = k_{\underline{d}}$ is an isomorphism of \mathbb{Z} -algebras.

Proof. By Lemma 6.11, it is straightforward to check that $e_{i,\underline{d}}$, $f_{i,\underline{d}}$ and $k_{\underline{d}}$ satisfy the relations $P_{ij,d}(0)$, $N_{ij,d}(0)$, and $C_{ij,d}(0)$. Thus η is well-defined. Also, Lemma 6.9 implies that the map is surjective. It remains to prove that η is injective.

We claim that, modulo the relations in $I_0(n, r)$, any path p in $\Sigma(n, r)$ is either equal to $k_{\underline{d}}$ or a path of the form

$$E(i_s, j_s, \underline{d}_s) \cdots E(i_1, j_1, \underline{d}_1) F(i'_1, j'_1, \underline{d}'_1) \cdots F(i'_t, j'_t, \underline{d}'_t),$$

satisfying the conditions in Lemma 6.10. Note that such a path is mapped onto one of monomial basis elements of Lemma 6.10, and so η is injective.

We prove the claim by induction on the length of p . If p has length less than or equal to one, it is equal to $k_{\underline{d}}$ or one of the arrows $F_{i,\underline{d}}$ and $E_{i,\underline{d}}$, and so the claim follows. Assume that p has length greater than one. Then we have

$$p = p' F_{i,\underline{c}} \text{ or } p = p' E_{i,\underline{c}}$$

where p' is a non-trivial path of smaller length, and so by induction has the required form

$$p' = EF = E(i_s, j_s, \underline{d}_s) \cdots E(i_1, j_1, \underline{d}_1) F(i'_1, j'_1, \underline{d}'_1) \cdots F(i'_t, j'_t, \underline{d}'_t),$$

where E and F are products of $E_{j,\underline{d}}$ and $F_{j,\underline{d}}$, respectively.

We first consider $p = p' E_{i,\underline{c}}$. If p' contains no $F_{j,\underline{d}}$, then the claim follows using the relations $P_{ab,\underline{d}}(0)$. Otherwise, by the relations $C_{ab,\underline{d}}(0)$, either $p = EF'$ with the length of F' smaller than that of F or $p = EE_{i,\underline{d}_1-\alpha_i+\alpha_{i+1}} F'$ with each factor $F(i'_l, j'_l, \underline{d}'_l)$ in F replaced with a factor $F(i'_l, j'_l, \underline{d}'_l)$. In the first case, the claim follows by induction. Otherwise, by the relations $P_{ab,\underline{d}}(0)$, there are two possibilities. First, there exists a minimal m with $j_m = i - 1$. Then $EE_{i,\underline{d}_1-\alpha_i+\alpha_{i+1}} F'$ is equal to

$$E(i_s, j_s, \underline{d}_s) \cdots E(i_{m-1}, j_{m-1}, \underline{d}_{m-1}) E(i_m, j_m + 1, \underline{c}_m) E(i_{m+1}, j_{m+1}, \underline{c}_{m+1}) \cdots E(i_1, j_1, \underline{c}_1) F'.$$

We have

$$c_1 = d_1 - \alpha_i + \alpha_{i+1} \geq \sum_l \alpha_{j_l+1} - \alpha_i + \alpha_{i+1} + \sum_l \alpha_{j'_l+1} = \sum_{l \neq m} \alpha_{j_l+1} + \alpha_{j_m+1} + \sum_l \alpha_{j'_l+1}.$$

Moreover, again using the relations $P_{ab,\underline{d}}(0)$, the factors can be reordered (up to change of d_l, c_m) to obtain a path of the required form.

Second, there is no such m with $j_m = i - 1$. Then

$$EE_{i,\underline{d}_1 - \alpha_i + \alpha_{i+1}} F' = E(i_s, j_s, \underline{d}_s) \cdots E(i_m, j_m, \underline{d}_m) E(i, i, \underline{c}_m) E(i_{m-1}, j_{m-1}, \underline{c}_{m-1}) \cdots E(i_1, j_1, \underline{c}_1) F',$$

with $j_{m-1} \leq i$ and $j_m > i$. In order to show that this path is of the required form, we need only to prove the inequality

$$c_1 = d_1 - \alpha_i + \alpha_{i+1} \geq \sum_l \alpha_{j_l+1} + \alpha_{i+1} + \sum_l \alpha_{j'_l+1}.$$

Clearly, the inequality holds for each component different from i . Since there are no m with $j_m = i - 1$, the sum $\sum_l \alpha_{j_l+1}$ contain no α_i . Since $FE_{i,c} = E_{i,c_1} F'$ with the length of F' equal to that of F , we must have $(d_1 - \alpha_i)_i \geq (\sum_l \alpha_{j'_l+1})_i$ and so the inequality follows.

Finally, we consider $p = p' F_{i,\underline{c}}$, where p' is a path of the required form $p' = EF$ as above. If there are no factor $E_{j,\underline{d}}$ in p' , then the claim follows from the relations $N_{ab,\underline{d}}(0)$. Otherwise $p = E' E_{j,\underline{d}_1} F$, which following $C_{ab,\underline{d}}(0)$ is either $p = E' F'$ with F' shorter than F , or $p = E' F' E_{j,\underline{c}}$ which is then the case showed above. So the claim holds. \square

7.2. A geometric realisation of $S_0(n, r)$. We now prove the main result of this section.

Theorem 7.3. *The map*

$$\psi : G(n, r) \rightarrow S_0(n, r)$$

defined by $\psi(e_{i,\underline{d}}) = e_{i,\underline{d}}$, $\psi(f_{i,\underline{d}}) = f_{i,\underline{d}}$ and $\psi(k_{\underline{d}}) = k_{\underline{d}}$ is an isomorphism of \mathbb{Z} -algebras.

Proof. From Proposition 5.5, we have the surjective \mathcal{Q} -algebra homomorphism

$$\mathcal{Q}\Sigma(n, r)/\mathcal{Q}I(n, r) \rightarrow S_q(n, r) \otimes_{\mathbb{Z}[q]} \mathcal{Q},$$

which, since $\mathcal{Q}/q\mathcal{Q} \simeq \mathbb{Z}$, induces a surjective \mathbb{Z} -algebra homomorphism

$$\mathcal{Q}\Sigma(n, r)/\mathcal{Q}I(n, r) \otimes_{\mathcal{Q}} \mathcal{Q}/q\mathcal{Q} \rightarrow S_q(n, r) \otimes_{\mathbb{Z}[q]} \mathcal{Q} \otimes_{\mathcal{Q}} \mathcal{Q}/q\mathcal{Q}.$$

Following the definition of $S_0(n, r)$ and the isomorphisms $\mathcal{Q}/q\mathcal{Q} \simeq \mathbb{Z}[q]/q\mathbb{Z}[q] \simeq \mathbb{Z}$, we have

$$S_q(n, r) \otimes_{\mathbb{Z}[q]} \mathcal{Q} \otimes_{\mathcal{Q}} \mathcal{Q}/q\mathcal{Q} = S_0(n, r)$$

and by Lemma 5.2

$$(\mathcal{Q}\Sigma(n, r)/\mathcal{Q}I(n, r)) \otimes_{\mathcal{Q}} \mathcal{Q}/q\mathcal{Q} \simeq (\mathbb{Z}\Sigma(n, r)/I_0(n, r)).$$

So there is a surjective \mathbb{Z} -algebra homomorphism $\mathbb{Z}\Sigma(n, r)/I_0(n, r) \rightarrow S_0(n, r)$ given by $E_{i,\underline{d}} \mapsto e_{i,\underline{d}}$, $F_{i,\underline{d}} \mapsto f_{i,\underline{d}}$, $K_{i,\underline{d}} \mapsto k_{i,\underline{d}}$. The theorem now follows from Theorem 7.2, since $G(n, r) = S_0(n, r)$ as \mathbb{Z} -modules. \square

Corollary 7.4. *The set $\psi(\mathcal{F} \times \mathcal{F}/\text{GL}(V))$ is a multiplicative basis for $S_0(n, r)$.*

7.3. Proof of Theorem 5.6. By Proposition 5.5, the map $\phi \otimes \mathcal{Q}$ induces a short exact sequence

$$0 \rightarrow K \rightarrow \mathcal{Q}\Sigma(n, r)/\mathcal{Q}I(n, r) \rightarrow S_q(n, r) \otimes_{\mathbb{Z}[q]} \mathcal{Q} \rightarrow 0.$$

Since $S_q(n, r) \otimes_{\mathbb{Z}[q]} \mathcal{Q}$ is a free \mathcal{Q} -module, applying $-\otimes_{\mathcal{Q}} \mathcal{Q}/q\mathcal{Q}$ gives the exact sequence

$$0 \rightarrow K \otimes_{\mathcal{Q}} \mathcal{Q}/q\mathcal{Q} \rightarrow \mathcal{Q}\Sigma(n, r)/\mathcal{Q}I(n, r) \otimes_{\mathcal{Q}} \mathcal{Q}/q\mathcal{Q} \rightarrow S_q(n, r) \otimes_{\mathbb{Z}[q]} \mathcal{Q} \otimes_{\mathcal{Q}} \mathcal{Q}/q\mathcal{Q} \rightarrow 0.$$

As in the proof of Theorem 7.3, we have isomorphisms $S_q(n, r) \otimes_{\mathbb{Z}[q]} \mathcal{Q} \otimes_{\mathcal{Q}} \mathcal{Q}/q\mathcal{Q} = S_0(n, r)$ and $(\mathcal{Q}\Sigma(n, r)/\mathcal{Q}I(n, r)) \otimes_{\mathcal{Q}} \mathcal{Q}/q\mathcal{Q} \simeq (\mathbb{Z}\Sigma(n, r)/I_0(n, r))$. Furthermore, via these two isomorphisms the map $\phi \otimes \mathcal{Q} \otimes \mathcal{Q}/q\mathcal{Q}$ is the composition of the isomorphism $\mathbb{Z}\Sigma(n, r)/I_0(n, r) \simeq G(n, r)$ in Theorem 7.2 and the isomorphism $G(n, r) \simeq S_0(n, r)$ in Theorem 7.3. Therefore $K \otimes_{\mathcal{Q}} \mathcal{Q}/q\mathcal{Q} = K/qK = 0$. Now by Nakayama's lemma there is an element $r = 1 + qf(q) \in \mathcal{Q}$ such that $rK = 0$. Since r is invertible in \mathcal{Q} , we have $K = 0$. Thus $\phi \otimes \mathcal{Q}$ is an isomorphism.

8. THE DEGENERATION ORDER ON PAIRS OF FLAGS

In this section let k be algebraically closed. We describe the degeneration order on $\mathrm{GL}(V)$ -orbits in $\mathcal{F} \times \mathcal{F}$ using quivers and the symmetric group S_r .

Let $\Gamma = \Gamma(n)$ be the quiver of type \mathbb{A}_{2n-1} ,

$$\Gamma : 1_L \longrightarrow 2_L \longrightarrow \cdots \longrightarrow n \longleftarrow \cdots \longleftarrow 2_R \longleftarrow 1_R$$

constructed by joining two linear quivers $\Lambda_L = \Lambda_L(n)$ and $\Lambda_R = \Lambda_R(n)$ at the vertex n . Often it will be clear from the context which side of Γ we are considering, and then we drop the subscripts on the vertices.

A pair $(f, f') \in \mathcal{F} \times \mathcal{F}$ is a representation of Γ , where f is supported on Λ_L , f' is supported on Λ_R . Conversely, any representation M of Γ with $\dim M_n = r$, which is projective when restricted to both Λ_L and Λ_R determines uniquely an orbit of pair of flags $[f, f'] \in \mathcal{F} \times \mathcal{F} / \mathrm{GL}(V)$. Moreover, two pairs of flags are isomorphic if and only if the corresponding representations are isomorphic.

For integers $i, j \in \{1, \dots, n\}$, let N_{ij} be the indecomposable representation of Γ which is equal to the indecomposable projective representations M_{in} and M_{jn} when restricted to Λ_L and Λ_R , respectively. A representation N of Γ which is projective when restricted to Λ_L and Λ_R , and $\dim N_n = r$, decomposes up to isomorphism as

$$N \simeq \bigoplus_{l=1}^r N_{i_l j_l}.$$

In this section we always assume that $j_1 \leq j_2 \leq \cdots \leq j_r$. The variety of representations that has such a decomposition is an open subset of the variety of representations of Γ with dimension vector the same as N . We shall view representations in this subvariety as pairs of flags in V .

Let \leq_{deg} denote the degeneration order on isomorphism classes of representations of Γ . That is, $M \leq_{deg} N$ for two representations M and N , if N is contained in the closure of the orbit of M in the space of all representations. The degeneration order on pairs of flags is also denoted by \leq_{deg} , since there is a degeneration between two pair of flags if and only if there is a degeneration between the corresponding representations of Γ .

Since Γ is a Dynkin quiver, by a result of Bongartz [2], the degeneration \leq_{deg} is the same as the degeneration \leq_{ext} given by a sequence of extensions. That is, if there is an extension

$$0 \longrightarrow N' \longrightarrow M \longrightarrow N'' \longrightarrow 0,$$

then $M \leq_{ext} N' \oplus N''$, and more generally \leq_{ext} is the transitive closure.

The symmetric group S_r of permutations of the set $\{1, \dots, r\}$ acts on representations with a decomposition $N = \bigoplus_{l=1}^r N_{i_l j_l}$ by

$$\sigma N = \bigoplus_{l=1}^r N_{i_{\sigma l} j_l}$$

for $\sigma \in S_r$.

The following facts are the key lemmas on degenerations in $\mathcal{F} \times \mathcal{F}$. For the sake of completeness we include a brief sketch of the proofs.

Lemma 8.1. *Let $N = \bigoplus_{l=1}^r N_{i_l j_l}$ be a decomposition as above, and let (t, s) with $t < s$ be a transposition. Then $N <_{deg} (t, s)N$ if and only if $i_t > i_s$.*

Proof. Assume that $i_t > i_s$. There is a short exact sequence

$$0 \longrightarrow N_{i_t j_s} \longrightarrow N_{i_t j_t} \oplus N_{i_s j_s} \longrightarrow N_{i_s j_t} \longrightarrow 0.$$

Since every extension degenerates to the trivial extension we have

$$N_{i_t j_t} \oplus N_{i_s j_s} \leq_{deg} N_{i_t j_s} \oplus N_{i_s j_t}$$

and therefore $N <_{deg} (t, s)N$.

Conversely, assume that $i_t \leq i_s$. By comparing the dimensions of the stabilisers of N and $(t, s)N$ we see that $N \not<_{deg} (t, s)N$. \square

We say that a degeneration $M \leq_{deg} N$ is minimal if $M \not\simeq N$ and $M \leq_{deg} X \leq_{deg} N$ implies $X \simeq M$ or $X \simeq N$.

Lemma 8.2. *Let $N = \bigoplus_{l=1}^r N_{i_l j_l}$ and $M \leq_{deg} N$ be minimal. Then there exists a transposition (t, s) such that $M \simeq (t, s)N$.*

Proof. Since $M \leq_{deg} N$ is minimal, there is a non-split extension

$$0 \longrightarrow N' \longrightarrow M \longrightarrow N'' \longrightarrow 0,$$

where $N \simeq N' \oplus N''$. We may choose summands $N_{i_s j_s}$ and $N_{i_t j_t}$ of N' and N'' , respectively, such that taking pushout along the projection $N' \rightarrow N_{i_s j_s}$ and then pullback along the inclusion $N_{i_t j_t} \rightarrow N''$ gives us a non-split extension

$$0 \longrightarrow N_{i_s j_s} \longrightarrow M' \longrightarrow N_{i_t j_t} \longrightarrow 0.$$

This extension is of the form of the extension in the proof of Lemma 8.1. Hence

$$M' <_{deg} N_{i_s j_s} \oplus N_{i_t j_t},$$

and so

$$M' \oplus (N'/N_{i_s j_s}) \oplus (N''/N_{i_t j_t}) <_{deg} (N'/N_{i_s j_s}) \oplus (N''/N_{i_t j_t}) \oplus N_{i_s j_s} \oplus N_{i_t j_t} \simeq N.$$

By the construction of M' ,

$$M \leq_{deg} M' \oplus (N'/N_{i_s j_s}) \oplus (N''/N_{i_t j_t}),$$

so by the minimality of the degeneration,

$$M \simeq M' \oplus (N'/N_{i_s j_s}) \oplus (N''/N_{i_t j_t}),$$

and so the lemma follows. \square

There is a unique closed orbit in $\mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{e}}$. We describe a corresponding representation.

Lemma 8.3. *The orbit of a pair of flags corresponding to a representation N is closed, if and only if $N \simeq \bigoplus_{l=1}^r N_{i_l j_l}$ with $i_l \leq i_{l+1}$ for all $l = 1, \dots, r-1$.*

Proof. A representation $N = \bigoplus_{l=1}^r N_{i_l j_l}$ with $i_l \leq i_{l+1}$ does not have degenerations according to Lemma 8.1. Hence N , and therefore also the corresponding pair of flags, has a closed orbit.

Conversely, if $i_l > i_{l+1}$, then N has a degeneration again by Lemma 8.1, and so the orbit of N is not closed. \square

Alternatively, we may prove the lemma by observing that among all representations of the form $N = \bigoplus_{l=1}^r N_{i_l j_l}$ the representation with $i_l \leq i_{l+1}$ has a stabiliser of maximal dimension, and so this representation has a closed orbit. The stabiliser in this case is a parabolic in $\mathrm{GL}(V)$.

There is a unique open orbit in $\mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{e}}$ with a corresponding representation given as follows. The proof is similar to the proof of the previous lemma.

Lemma 8.4. *The orbit of a pair of flags corresponding to a representation N is open, if and only if $N \simeq \bigoplus_{l=1}^r N_{i_l j_l}$ with $i_l \geq i_{l+1}$ for all $l = 1, \dots, r-1$.*

Similar to the closed orbit, a representation of the form $N \simeq \bigoplus_{l=1}^r N_{i_l j_l}$ with $i_l \geq i_{l+1}$ has a stabiliser of minimal dimension, and so the orbit is open. The stabiliser in this case is the intersection of two opposite parabolics in $\mathrm{GL}(V)$. Such stabilisers are called biparabolic or seaweeds [5]. The stabiliser of an arbitrary pair of flags is equal to the intersection of two parabolics in $\mathrm{GL}(V)$.

Let $o_{\underline{d}, \underline{e}}$ denote the unique open orbit and $k_{\underline{d}, \underline{e}}$ the unique closed orbit in $\mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{e}}$. Then $k_{\underline{d}} = k_{\underline{d}, \underline{d}}$ and we let $o_{\underline{d}} = o_{\underline{d}, \underline{d}}$. Let $\tau \in S_r$, then $\tau o_{\underline{d}, \underline{e}} \in \mathcal{F} \times \mathcal{F}/\mathrm{GL}(V)$ denotes the orbit of pairs of flag corresponding to the representation τN , where $N = \bigoplus_{l=1}^r N_{i_l j_l}$ with $i_{l+1} \leq i_l$, $\underline{d} = i_1 + \dots + i_n$ and $\underline{e} = j_1 + \dots + j_n$. Similarly, we let $\tau k_{\underline{d}, \underline{e}}$ be defined as the orbit corresponding to τN , where $N = \bigoplus_{l=1}^r N_{i_l j_l}$ with $i_{l+1} \geq i_l$.

9. IDEMPOTENTS FROM OPEN ORBITS

Let $M(n, r)$ be the submodule of $G(n, r)$ with basis the open orbits in $\mathcal{F} \times \mathcal{F}$. In this section we prove that $M(n, r)$ is a subalgebra $G(n, r)$ which is also a direct factor. We also show that $M(n, r)$ is isomorphic to the \mathbb{Z} -algebra of $|D(n, r)| \times |D(n, r)|$ -matrices, where $|D(n, r)|$ is the number of decompositions of r into n parts.

We start with two lemmas relating degeneration and multiplication in $G(n, r)$. Let \leq_{deg} be the degeneration order on orbits in $(\mathcal{F} \times \mathcal{F}) \times (\mathcal{F} \times \mathcal{F})$ with the action of $\mathrm{GL}(V) \times \mathrm{GL}(V)$.

Lemma 9.1. *If $e_B \times e_{B'} \leq_{deg} e_A \times e_{A'}$, then $e_B \star e_{B'} \leq_{deg} e_A \star e_{A'}$*

Proof. Since $e_B \times e_{B'} \subseteq \overline{e_A \times e_{A'}}$ we have $S(B, B') \subseteq \overline{S(A, A')}$. By Corollary 6.2, we have that $\overline{S(A, A')}$ is the orbit closure of $e_A \star e_{A'}$ and $\overline{S(B, B')}$ is the orbit closure of $e_B \star e_{B'}$, the lemma follows. \square

We have the following key lemma on degeneration and multiplication in $G(n, r)$.

Lemma 9.2. *Let $\sigma \in S_r$, $e_{B'} \subseteq \mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{e}}$ and $e_B \subseteq \mathcal{F}_{\underline{f}} \times \mathcal{F}_{\underline{g}}$. Then $e_{B'} \star (\sigma o_{\underline{e}, \underline{f}}) \star e_B \leq_{deg} \sigma o_{\underline{d}, \underline{g}}$.*

Proof. By Lemma 9.1, it suffices to consider the case where e_B and $e_{B'}$ are closed orbits. By Lemma 8.3, we may choose the representation

$$\bigoplus_{l=1}^r N_{j_l k_l},$$

where $k_{l+1} \geq k_l$ and $j_{l+1} \geq j_l$ for the orbit e_B . Similarly, $o_{\underline{e}, \underline{f}}$ is the orbit corresponding to the representation

$$\bigoplus_{l=1}^r N_{i_l j_l},$$

where $i_l \geq i_{l+1}$ by Lemma 8.4. Then the coefficient of $\sigma o_{\underline{e}, \underline{g}}$ in the product $(\sigma o_{\underline{e}, \underline{f}}) \cdot e_B$ in $S_q(n, r)$ is non-zero, and so

$$(\sigma o_{\underline{e}, \underline{f}}) \star e_B \leq_{deg} \sigma o_{\underline{e}, \underline{g}}.$$

Similarly,

$$e_{B'} \star \sigma o_{\underline{e}, \underline{g}} \leq_{deg} \sigma o_{\underline{d}, \underline{g}}.$$

By Lemma 9.1,

$$e_{B'} \star \sigma o_{\underline{e}, \underline{f}} \star e_B \leq_{deg} e_{B'} \star \sigma o_{\underline{e}, \underline{g}} \leq_{deg} \sigma o_{\underline{d}, \underline{g}},$$

as required. \square

Corollary 9.3. *Let $\sigma \in S_r$, $e_{B'} \subseteq \mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{e}}$ and $e_B \subseteq \mathcal{F}_{\underline{f}} \times \mathcal{F}_{\underline{g}}$. Then $e_{B'} \star (\sigma k_{\underline{e}, \underline{f}}) \star e_B \leq_{deg} \sigma k_{\underline{d}, \underline{g}}$.*

Proof. The corollary follows from the previous lemma since $\sigma k_{\underline{d}, \underline{g}} = \sigma \iota o_{\underline{d}, \underline{e}}$, where $\iota(i) = n - i + 1$. \square

Corollary 9.4. *Let $e_{B'} \subseteq \mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{e}}$ and $e_B \subseteq \mathcal{F}_{\underline{f}} \times \mathcal{F}_{\underline{g}}$. Then $e_{B'} \star o_{\underline{e}, \underline{f}} \star e_B = o_{\underline{d}, \underline{g}}$. In particular, $o_{\underline{d}, \underline{e}} \star o_{\underline{e}, \underline{f}} = o_{\underline{d}, \underline{f}}$.*

Proof. By the lemma we know that $e_{B'} \star o_{\underline{e}, \underline{f}} \star e_B \leq_{deg} o_{\underline{d}, \underline{g}}$. Since $o_{\underline{d}, \underline{g}}$ is the unique dense open orbit in $\mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{g}}$, the equality follows. \square

Corollary 9.5. *$M(n, r)$ is a two-sided ideal in $G(n, r)$.*

Proof. The previous corollary shows that the \mathbb{Z} -submodule $M(n, r) \subseteq G(n, r)$ is closed under multiplication from both sides with elements from $G(n, r)$, and so it is a two-sided ideal. \square

We will now show that $M(n, r)$ is a direct factor of $G(n, r)$.

Lemma 9.6. *$\{o_{\underline{d}}\}_{\underline{d}} \cup \{k_{\underline{d}} - o_{\underline{d}}\}_{\underline{d}}$ is a set of pairwise orthogonal idempotents in $G(n, r)$.*

Proof. By Corollary 9.4, $(o_{\underline{d}})^2 = o_{\underline{d}}$, $(k_{\underline{d}} - o_{\underline{d}})o_{\underline{d}} = o_{\underline{d}} - o_{\underline{d}} = 0$, $o_{\underline{d}}(k_{\underline{d}} - o_{\underline{d}}) = o_{\underline{d}} - o_{\underline{d}} = 0$, and $(k_{\underline{d}} - o_{\underline{d}})^2 = (k_{\underline{d}} - o_{\underline{d}} - o_{\underline{d}} + o_{\underline{d}}) = k_{\underline{d}} - o_{\underline{d}}$. All other orthogonality relations follow from the definition of multiplication in $S_q(n, r)$. \square

Let $M(D(n, r))$ be the algebra of $|D(n, r)| \times |D(n, r)|$ -matrices over \mathbb{Z} . Let

$$\omega_0 : M(n, r) \rightarrow M(D(n, r))$$

be the \mathbb{Z} -linear map where $\omega_0(o_{\underline{d}, \underline{e}}) = E_{\underline{d}, \underline{e}}$ is the $(\underline{d}, \underline{e})$ 'th elementary matrix in $M(D(n, r))$.

Lemma 9.7. *The map $\omega_0 : M(n, r) \rightarrow M(D(n, r))$ is a \mathbb{Z} -algebra isomorphism.*

Proof. The proposition is an immediate consequence of Lemma 9.6 and Corollary 9.4. \square

We construct a section to the inclusion $M(n, r) \subseteq G(n, r)$.

Lemma 9.8. *The map $\omega : G(n, r) \rightarrow M(n, r)$ defined by*

$$\omega(e_A) = o_{\underline{d}, \underline{e}}$$

for all $e_A \subseteq \mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{e}}$ is a surjective \mathbb{Z} -algebra homomorphism.

Proof. The map is clearly a surjective \mathbb{Z} -module homomorphism. Let $e_A \subseteq \mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{e}}$ and $e_B \subseteq \mathcal{F}_{\underline{e}} \times \mathcal{F}_{\underline{f}}$. Then $\omega(e_A \star e_B)$ is the unique open orbit in $\mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{f}}$, which is equal to $\omega(e_A) \star \omega(e_B)$, by Corollary 9.4. Moreover, $\omega(1_{G(n,r)}) = \omega(\sum_{\underline{d}} k_{\underline{d}}) = \sum_{\underline{d}} o_{\underline{d}} = 1_{M(n,r)}$. This completes the proof of the lemma. \square

We can now prove the main result of this section.

Theorem 9.9. *We have an isomorphism of \mathbb{Z} -algebras $G(n, r) \rightarrow M(n, r) \times (G(n, r)/M(n, r))$ given by $e_A \mapsto (\omega(e_A), \overline{e_A})$.*

Proof. By Corollary 9.4, we have

$$M(n, r) = (\sum_{\underline{d}} o_{\underline{d}}) G(n, r) (\sum_{\underline{d}} o_{\underline{d}}).$$

Now, $1_{G(n,r)} = \sum_{\underline{d}} k_{\underline{d}}$, and again by Corollary 9.4, $\sum_{\underline{d}} o_{\underline{d}}$ is a central idempotent in $G(n, r)$. This proves that $M(n, r)$ is a direct factor in $G(n, r)$, and so the theorem follows. \square

Let $\tilde{\mathbb{A}}_n$ denote the preprojective algebra of type \mathbb{A}_n . See [4] for the definition and properties of preprojective algebras.

Corollary 9.10. $S_0(2, r) \simeq M(2, r) \times \tilde{\mathbb{A}}_{r-1}$

Proof. We need to show that

$$(\sum_{\underline{d}} k_{\underline{d}} - o_{\underline{d}}) G(n, r) (\sum_{\underline{d}} k_{\underline{d}} - o_{\underline{d}}) \simeq \tilde{\mathbb{A}}_{r-1}.$$

First observe that $(\sum_{\underline{d}} k_{\underline{d}} - o_{\underline{d}}) G(n, r) (\sum_{\underline{d}} k_{\underline{d}} - o_{\underline{d}})$ is generated by $e_{1,\underline{d}} - o_{\underline{d}-\alpha_2+\alpha_1,\underline{d}}$, $f_{1,\underline{d}} - o_{\underline{d}+\alpha_2-\alpha_1,\underline{d}}$ and $k_{\underline{d}} - o_{\underline{d}}$. A direct computation shows that the generators satisfy the preprojective relations. By comparing dimensions we get the required isomorphism. \square

Let $\underline{n} = n_1 + \dots + n_l$ and $\underline{r} = r_1 + \dots + r_l$ be decompositions of n and r , respectively, into l parts, where $n_i > 0$. Let $m_i = \sum_{j=1}^{i-1} n_j$, where $m_1 = 0$.

There is a map ϕ_j of flags of length n_j to flags of length n given by $\phi_j(f)_l = 0$ for $l \leq m_j$, $\phi_j(f)_l = f_{l-m_j}$ for $m_j < l \leq m_{j+1}$ and $\phi_j(f)_l = f_{n_j}$ for $l > m_{j+1}$. The corresponding map on orbits of pairs of flags

$$[f, f'] \mapsto [\phi_j(f), \phi_j(f')]$$

is also denoted by ϕ_j .

Let

$$\phi_{\underline{n}, \underline{r}} : G(n_1, r_1) \times \dots \times G(n_l, r_l) \rightarrow G(n, r)$$

be the \mathbb{Z} -linear map defined by

$$(N_1, \dots, N_l) \mapsto \phi_1(N_1) \oplus \dots \oplus \phi_l(N_l).$$

Lemma 9.11. *The map*

$$\phi_{\underline{n}, \underline{r}} : G(n_1, r_1) \times \dots \times G(n_l, r_l) \rightarrow G(n, r)$$

is an injective \mathbb{Z} -algebra homomorphism. Moreover, $\phi_{\underline{n}, \underline{r}}(N_1, \dots, N_l) \leq_{deg} \phi_{\underline{n}, \underline{r}}(N'_1, \dots, N'_l)$ if and only if $N_i \leq_{deg} N'_i$ for all i .

Proof. Since $\phi_{\underline{n}, \underline{r}}$ is injective on basis elements, it is an injective \mathbb{Z} -linear map. By Lemma 2.2, in terms of matrices, the map is given by

$$\phi_{\underline{n}, \underline{r}}(e_{A_1}, \dots, e_{A_l}) = e_{A_1 \oplus \dots \oplus A_l}.$$

Following Lemma 6.11, the map $\phi_{\underline{n}, \underline{r}}$ preserves multiplication and thus is an injective \mathbb{Z} -algebra homomorphism.

Let

$$N = \phi_{\underline{n}, \underline{r}}(N_1, \dots, N_l) \text{ and } N' = \phi_{\underline{n}, \underline{r}}(N'_1, \dots, N'_l),$$

and $N \leq_{deg} N'$. We may assume that the degeneration is minimal. By Lemma 8.2, $N' = (t, s)N$ for a transposition (t, s) . Then the transposition (t, s) must act within one N_i , since the off-diagonal blocks of the matrices of both N and N' are zero, and so

$$(t, s)N = \phi_{\underline{n}, \underline{r}}(N_1, \dots, N_{i-1}, (t', s')N_i, N_{i+1}, \dots, N_l)$$

for a transposition (t', s') . This shows that $N_i \leq_{deg} N'_i$ for all i .

The converse also follows from Lemma 8.2. \square

Let \underline{n} , \underline{r} and m_i be as above. Let $\underline{r}_i = d_{m_i+1} + \cdots + d_{m_{i+1}}$, a decomposition of r_i into n_i parts, and $\underline{d} = d_1 + \cdots + d_n$, which is an element in $D(n, r)$. Let

$$o_{(\underline{d}, \underline{n})} = \phi(o_{\underline{r}_1}, \dots, o_{\underline{r}_l}),$$

which is an idempotent, by Lemma 9.11. Note that $k_{\underline{d}} = o_{(\underline{d}, 1+\dots+1)}$ and that $o_{\underline{d}} = o_{(\underline{d}, n)}$, where n denotes the trivial decomposition of n into 1 part.

For a given \underline{d} , there is one idempotent for each decomposition \underline{n} , and so this produces 2^{n-1} idempotents in $k_{\underline{d}}G(n, r)k_{\underline{d}}$, if $k_{\underline{d}}$ is in the interior of the quiver of $G(n, r)$ viewed as an $(n-1)$ -simplex. If $k_{\underline{d}}$ is on the boundary, but in the interior of a t -simplex, then there are 2^t idempotents. In particular, on a line we get the two idempotents $k_{\underline{d}}$ and $o_{\underline{d}}$, and for the vertices of the simplex we have the unique idempotent $k_{\underline{d}} = o_{\underline{d}}$.

Lemma 9.12. *If $o_{(\underline{d}, \underline{n})} \leq_{deg} N$, then $o_{(\underline{d}, \underline{n})} \star N = N \star o_{(\underline{d}, \underline{n})} = o_{(\underline{d}, \underline{n})}$.*

Proof. We have $o_{(\underline{d}, \underline{n})} = o_{(\underline{d}, \underline{n})} \star o_{(\underline{d}, \underline{n})} \leq_{deg} N \star o_{(\underline{d}, \underline{n})} \leq_{deg} k_{\underline{d}} \star o_{(\underline{d}, \underline{n})} = o_{(\underline{d}, \underline{n})}$ by Lemma 9.1. The proof of the other equality is similar. \square

10. GEOMETRIC REALISATION OF 0-HECKE ALGEBRAS

In this section let $n = r$ and $\underline{d} = 1 + \cdots + 1$. In this case $\mathcal{F}_{\underline{d}}$ is the complete flag variety. The idempotent $k_{\underline{d}}$ is then the unique interior vertex in the quiver of $G(n, n)$. Let $H_0(n) = k_{\underline{d}}G(n, n)k_{\underline{d}}$. The \mathbb{Z} -algebra $H_0(n)$ is called the 0-Hecke algebra [3, 9, 20, 19], and we give a proof of this fact in this section.

From the previous section we have 2^{n-1} distinct idempotents $o_{(\underline{d}, \underline{n})}$, one for each decomposition $\underline{n} = n_1 + \cdots + n_l$ of n with $n_i > 0$.

Let

$$t_i = (i, i+1)k_{\underline{d}}.$$

We have

$$t_i = o_{(\underline{d}, \underline{n})},$$

where $n = n_1 + \cdots + n_{r-1}$ with $n_i = 2$ and $n_j = 1$ for $j \neq i$, and so t_i is an idempotent. Also

$$t_i = f_{\underline{d}+\alpha_i-\alpha_{i+1}} \star e_{i, \underline{d}} = e_{\underline{d}-\alpha_i+\alpha_{i+1}} \star f_{i, \underline{d}}.$$

For a permutation σ and a transposition $(i, i+1)$, define

$$(i, i+1) \star \sigma = \begin{cases} (i, i+1)\sigma & \text{if } (i, i+1)\sigma \leq_{deg} \sigma \text{ and} \\ \sigma & \text{otherwise.} \end{cases}$$

Write $\tau \leq_{\star} \sigma$ if $\tau = (i, i+1) \star \sigma$ for some i , and denote the closure as a partial order also by \leq_{\star} . We clearly have that $\tau \leq_{\star} \sigma$ implies $\tau \leq_{deg} \sigma$.

Lemma 10.1. $t_i \star \sigma k_{\underline{d}} = ((i, i+1) \star \sigma)k_{\underline{d}}$

Proof. Let A be a matrix such that $e_A = \sigma k_{\underline{d}}$. The matrix A is a permutation matrix. We assume that $A_{i,r} = A_{i+1,s} = 1$. By Lemma 6.11,

$$t_i \star \sigma k_{\underline{d}} = f_{i, \underline{d}+\alpha_i-\alpha_{i+1}} \star e_{i, \underline{d}} \star e_A = \begin{cases} e_A & \text{if } r > s \\ e_{A'} & \text{if } r < s, \end{cases}$$

where A' is obtained from A by swapping the i 'th and $(i+1)$ 'th rows. The lemma follows since $e_{A'} = (i, i+1)\sigma k_{\underline{d}}$ and since $r < s$ if and only if $(i, i+1)\sigma \leq_{deg} \sigma$. \square

Although it can be deduced from a bubble sort algorithm that the t_i generate $H_0(n)$ as an algebra, we will give an explicit construction of each of the basis element using the multiplication in $H_0(n)$.

Lemma 10.2. *Suppose $i < j$. Then $t_i \star t_{i+1} \star \cdots \star t_{j-1} = (i, i+1, \dots, j)k_{\underline{d}}$, where $(i, i+1, \dots, j)$ is a cycle in S_n .*

Proof. The corollary follows from the previous lemma by induction, since $(i, i+1, \dots, j) = (i, i+1)(i+1, \dots, j) = (i, i+1) \star (i+1, \dots, j)$. \square

Let σ be a permutation. Let

$$t^\sigma = t^{\sigma, n} \star \cdots \star t^{\sigma, 1}$$

be defined by

$$t^{\sigma, 1} = t_1 \star t_2 \star \cdots \star t_{\sigma^{-1}(1)-1}$$

and then

$$t^{\sigma, i} = t_i \star t_{i+1} \star \cdots \star t_{\tau_{i-1}\sigma^{-1}(i)-1},$$

where τ_{i-1} is given by

$$\tau_{i-1}k_{\underline{d}} = t^{\sigma, i-1} \star \cdots \star t^{\sigma, 1}.$$

Theorem 10.3. *With the notation above, $t^\sigma = \sigma k_{\underline{d}}$. Consequently, the set of all t^σ for $\sigma \in S_n$ is a multiplicative basis of $H_0(n)$.*

Proof. By the previous lemma, $t^{\sigma, 1} = (1, \dots, \sigma^{-1}(1))k_{\underline{d}}$. As a representation $\tau_1 k_{\underline{d}} = t^{\sigma, 1}$ has the summand $N_{1, \sigma^{-1}(1)}$, which is fixed by any t_i for $i > 1$, and therefore by $t^{\sigma, i}$ for $i > 1$. By induction $\tau_i k_{\underline{d}}$ has the summands $N_{j, \sigma^{-1}(j)}$ for $j = 1, \dots, i$, which are fixed by $t^{\sigma, j}$ for $j > i$. Therefore $t^\sigma = \sigma k_{\underline{d}}$. \square

We construct the idempotents $o_{(\underline{d}, \underline{n})}$ using the generators t_i . Let $[i, j]$ be an interval in $[1, \dots, n]$. Define $t^{[i, j]}$ by induction as follows. Let $t^{[i, i]} = k_d$ and

$$t^{[i, j]} = t^{[i+1, j]} \star t_i \star \cdots \star t_{j-1}.$$

To each decomposition $\underline{n} = n_1 + \cdots + n_l$ we have the element

$$t^{\underline{n}} = t^{[m_1+1, m_2]} \star \cdots \star t^{[m_l+1, m_{l+1}]}$$

where $m_i = \sum_{j=1}^{i-1} n_j$ and $m_1 = 0$.

Corollary 10.4. $t^{\underline{n}} = O_{(\underline{d}, \underline{n})}$.

We end this section by giving a proof of the fact that $H_0(n)$ is the 0-Hecke algebra given with generators and relations for instance in [19]. Now recall the 0-Hecke algebra, denoted by $\mathcal{H}_0(n)$, which is an \mathbb{C} -algebra generated by \mathcal{T}_i for $i = 1, \dots, n-1$ with generating relations

- i) $\mathcal{T}_i^2 = -\mathcal{T}_i$,
- ii) $\mathcal{T}_i \mathcal{T}_{i+1} \mathcal{T}_i = \mathcal{T}_{i+1} \mathcal{T}_i \mathcal{T}_{i+1}$ and
- iii) $\mathcal{T}_i \mathcal{T}_j = \mathcal{T}_j \mathcal{T}_i$ for $|i - j| > 1$.

The algebra $\mathcal{H}_0(n)$ is a specialisation of the q -Hecke algebra at $q = 0$ and has dimension $n!$.

Theorem 10.5. $H_0(n) \otimes_{\mathbb{Z}} \mathbb{C} \simeq \mathcal{H}_0(n)$

Proof. Let

$$h : \mathcal{H}_0(n) \rightarrow H_0(n) \otimes_{\mathbb{Z}} \mathbb{C}$$

be given by $h(\mathcal{T}_i) = -t_i$. A direct computation in $H_0(n)$ shows that $-t_i$ satisfy the 0-Hecke relations i), ii) and iii) above, so the map is well defined. The two algebras have the same dimension over \mathbb{C} , and so it suffices to know that the map is surjective. But the t_i are generators of $H_0(n)$ by Theorem 10.3 and so the proof is complete. \square

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